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AN OPERATOR THEORY OF PARAMETRIC PROGRAMMING FOR THE TRANSPORTATION PROBLEM – I*

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ABSTRACT

This paper investigates the effect on the optimum solution of a (capacitated) transportation problem when the data of the problem (the rim conditions – i.e., the warehouse supplies and market demands – the per unit transportation costs and the upper bounds) are continuously varied as a (linear) function of a single parameter. An operator theory is developed and algorithms provided for applying rim and cost operators that effect the transformation of optimum solution associated with changes in rim conditions and unit costs. Bound operators that effect changes in upper bounds are shown to be equivalent to rim operators. The discussion in this paper is limited to basis preserving operators for which the changes in the data are such that the optimum basis structures are preserved.

1. INTRODUCTION

One of the earliest and most fruitful applications of linear programming techniques has been the formulation and solution of the transportation (distribution) problem. Many practical problems having nothing to do with distribution also fit into this mathematical model for minimizing the total cost of transporting a homogenous good from a set of warehouses to a set of markets.

The term "parametric programming" when applied to the transportation problem refers to a study of the changes in the optimal solution when the data of the problem (the rim conditions – i.e., the warehouse supplies and market demands – and the per unit transportation costs) are *continuously* varied as a (linear) function of a parameter. We develop an "Operator Theory" to carry out the transformation of an optimal solution when there are changes in the data of the problem. We consider two general kinds of operators. Rim operators transform optimum solutions corresponding to changes in rim conditions and cost operators effect such transformations when the unit transportation costs are changed.

Such an analysis proves valuable in incorporating certain aspects of a problem already not captured by its mathematical formulation as a transportation model. For instance, in the formulation of a cash-management problem as a transshipment model [16], the optimum level of cash balance can be determined by using such techniques. The same approach is also useful in solving some operations research model types that cannot be formulated as a transportation model *per se*. For instance, it is well known

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[7, 14] that the traveling salesman problem can be solved by solving a sequence of assignment problems and improvements on such algorithms can readily be obtained by the use of operator theory [19].

In this paper we focus our attention on basis-preserving operators that effect data changes small enough not to alter the optimal "basis structure." In the sequel [20] we extend the results for data variations that can be arbitrarily large, and suggest procedures for post-optimization, i.e., finding the new optimal solution when a *discrete* change is made in the data of the problem. In that paper we also give economic and managerial interpretations of the operators and demonstrate their use in finding *real* shadow prices (i.e., change in optimal cost associated with nonzero changes in the data of the problem).

A note on methodology is in order. Throughout the paper our emphasis is on constructive procedures and proofs since they lead directly to usable and hopefully efficient computer programs. Our development of operator theory is straightforward and requires no reference to the parametric programming theory for the general linear program [9, 10] for justification. Our goal in this paper is to provide, simultaneously, new theoretical insights and to give constructive procedures for obtaining the practical benefits of the new results.

2. PRELIMINARY DEFINITIONS AND RESULTS

We shall consider the transportation problem with its classical interpretation as a problem of shipping at minimum total cost a homogenous good from m warehouses to n markets. Define the index sets:

$$I=\{1, 2, \ldots, m\}, \text{ the set of warehouses, and} \tag{1}$$

$$J=\{1, 2, \ldots, n\}, \text{ the set of markets.} \tag{2}$$

For $i\epsilon I$ and $j\epsilon J$ define

x_{ij} = amount of the goods shipped from warehouse i to market j,
c_{ij} = cost of shipping one unit of the goods from warehouse i to market j,
U_{ij} = maximum amount that can be shipped from warehouse i to market j,
a_i = availability or supply at warehouse i, and
b_j = requirement or demand at market j.

We will refer to c_{ij}, U_{ij}, a_i, and b_j as the data of the problem. In particular the a_i's and b_j's will be referred to as *rim conditions*. Then we shall define the (capacitated) transportation problem [6, 8, 15] P as:

$$\text{Minimize} \sum_{i\epsilon I} \sum_{j\epsilon J} c_{ij}x_{ij}=Z, \tag{3}$$

subject to the following constraints:

$$\sum_{j\epsilon J} x_{ij}=a_i \qquad \text{for } i\epsilon I, \tag{4}$$

$$\sum_{i\epsilon I} x_{ij}=b_j \qquad \text{for } j\epsilon J, \text{ and} \tag{5}$$

$$0\leq x_{ij}\leq U_{ij} \qquad \text{for } i\epsilon I \text{ and } j\epsilon J \tag{6}$$

and under the assumptions:

(A1) a_i, b_j, and U_{ij} are nonnegative numbers.

(A2) The system (4)–(6) has a feasible solution.

REMARK 1: Since assumption (A2) holds, we can sum the Equations (4) for $i \epsilon I$ and sum (5) for $j \epsilon J$ to get

$$\sum_{i \epsilon I} a_i = \sum_{i \epsilon I} \sum_{j \epsilon J} x_{ij} = \sum_{j \epsilon J} \sum_{i \epsilon I} x_{ij} = \sum_{j \epsilon J} b_j,$$

so that assumption (A2) implies:

(A3)
$$\sum_{i \epsilon I} a_i = \sum_{j \epsilon J} b_j.$$

Similarly the conditions $\sum_{j \epsilon J} U_{ij} \geqslant a_i$ for $i \epsilon I$ and $\sum_{i \epsilon I} U_{ij} \geqslant b_j$ for $j \epsilon J$ can be easily verified to be necessary for (A2) to hold.

For problems in which some or all of the x_{ij} are not bounded from above, we can take $U_{ij} = N$, where N is an arbitrarily large positive number.

There arise transportation problems for which the assumption (A3) is replaced by $\sum_{i \epsilon I} a_i \geqslant \sum_{j \epsilon J} b_j$ with the equality signs in (4) replaced by $\leqslant$ inequalities. Such problems can be brought to the standard form (3)–(6) by adding a dummy market [6, 11, 15, 20]. Problems with $\sum_{i \epsilon I} a_i \leqslant \sum_{j \epsilon J} b_j$ can be handled in a similar manner. Many other distribution problems can also be transformed to the problem P [4, 5, 13, 15, 18, 22].

By a cell (i, j) we mean an ordered index pair with *row* (warehouse) $i \epsilon I$ and *column* (market) $j \epsilon J$. A *line* refers to a row or column.

DEFINITION 1: Let Ω denote a collection of cells (i, j). *Line* g is said to be *connected* to line h in Ω if and only if there exists a *path* (sequence) S of distinct cells belonging to Ω;

(7)
$$S = \{(i_1, j_1), (i_2, j_2), \ldots\ldots, (i_k, j_k)\},$$

such that

(a) (i_1, j_1) is the only cell of S in line g and (i_k, j_k) is the only cell of S in line h,

(b) for every $1 < t < k$ either (i) or (ii) below holds:

(i) $i_t = i_{t-1}$ and $j_t = j_{t+1}$

(ii) $j_t = j_{t-1}$ and $i_t = i_{t+1}$.

EXAMPLE: In the tableau of Fig. 1(a) we have 12 cells (i, j) with row (warehouse) $i = 1,2,3$ and column (market) $j = 1,2,3,4$. If we denote by Ω the circled cells $\{(1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 4)\}$, row 3 is connected to column 2 in Ω by the sequences $S = \{(3,1), (1,1), (1,3), (2,3), (2,2)\}$, and column 2 is connected to column 3 in Ω by $S = \{(2, 2), (2, 3)\}$.

REMARK 2: From properties (a) and (b) of Definition 1, it follows that the number of cells in the path (7) connecting line g and line h is even if both g and h are rows (columns); it is odd if g is a row with h as a column or vice versa.

DEFINITION 2: A *basis* B is a set of $m + n - 1$ cells (called basic cells) such that every pair of lines is (uniquely) connected [15]. The addition of a nonbasic cell (p, q) to a basis B creates an unique *cycle*

(stepping-stone tour [2]) $\Gamma=\{(p, q)\} \cup S$ where S is the path in B connecting row p to column q. Equivalently, a basis B is a set of $m+n-1$ cells without any cycles [15].

REMARK 3: From Remark 2 and Definition 2, it follows that the number of cells in a cycle is even (say $k=2l$) and we are permitted to make the following definition.

DEFINITION 3: Let the cycle Γ be given by

$$\Gamma=\{(p, q)=(i_0, j_0)\} \cup \{(i_1, j_1), (i_2, j_2), \ldots, (i_{2l-2}, j_{2l-2}), (i_{2l-1}, j_{2l-1})\}. \tag{8}$$

Let

$$\Gamma_1=\{(i_1, j_1), (i_3, j_3), \ldots, (i_{2l-1}, j_{2l-1})\} \text{ and}$$

$$\Gamma_2=\{(i_0, j_0), (i_2, j_2), \ldots, (i_{2l-2}, j_{2l-2})\}.$$

We define Γ_1 as the cells with *odd parity* and Γ_2 as the cells with *even parity*, each set having exactly l members.

REMARK 4: From Definitions 1, 2, and 3, it follows that every row (column) has either two cells of Γ (one with even and another with odd parity) or no cells at all.

EXAMPLE (cont'd): The reader may verify that the circled cells of Fig. 1(a) constitute a basis. When the nonbasic cell (3, 2) is added to this basis, the cycle $\Gamma=\{(3, 2), (3, 1), (1, 1), (1, 3), (2, 3), (2, 2)\}$ results (Fig. 1(c)) with $\Gamma_1=\{(3, 1), (1, 3), (2, 2)\}$, and $\Gamma_2=\{(3, 2), (1, 1), (2, 3)\}$. Remarks 3 and 4 may be readily verified for this cycle.

DEFINITION 4: A *basic solution* $X=\{x_{ij}\}$ corresponding to a basis B satisfies (4) and (5) and is such that $(i, j) \notin B$ implies that either $x_{ij}=0$ or $x_{ij}=U_{ij}$. If, in addition, (6) is satisfied for $(i, j) \in B$, the basic solution is called *primal feasible*. We define LB and UB as the sets of nonbasic variables that are at their lower and upper bounds, respectively. We define (B, LB, UB) as a *basis structure*. It is well known [6, 15] that given a problem and a basis structure, the associated primal solution is unique.

We now define the *dual problem* to P. Let u_i for $i \epsilon I$, v_j for $j \epsilon J$ and w_{ij} for $i \epsilon I$ and $j \epsilon J$ be dual variables associated with the row constraints (4), column constraints (5), and upperbound constraints (6), respectively. Then the dual problem is

$$\text{Maximize} \sum_{i \epsilon I} a_i u_i + \sum_{j \epsilon J} b_j v_j - \sum_{i \epsilon I} \sum_{j \epsilon J} U_{ij} w_{ij} = F, \tag{9}$$

subject to the following constraints:

$$u_i + v_j - w_{ij} \leqslant c_{ij} \qquad \text{for } i \epsilon I \text{ and } j \epsilon J \tag{10}$$

$$w_{ij} \geqslant 0 \qquad \text{for } i \epsilon I \text{ and } j \epsilon J. \tag{11}$$

Given a basis B, one can determine a one-parameter family of solutions u_i and v_j to the equations

$$u_i + v_j = c_{ij} \qquad \text{for } (i, j) \epsilon B, \tag{12}$$

such that $u_i + v_j = d_{ij}$ is unique for all $i \epsilon I$ and $j \epsilon J$ [6, 11, 15]. The above reference also provide an algorithm for determining these variables. We call $D=\{d_{ij}\}$ as the *dual matrix* since the d_{ij}'s are uniquely determined though u_i and v_j are determined only up to an additive constant.

It is well known [6, 15] that a primal basic feasible solution is *optimal* if its dual solutions satisfy

$$u_i + v_j \leq c_{ij} \qquad \text{for } (i, j) \in LB, \text{ and} \tag{13}$$

$$u_i + v_j \geq c_{ij} \qquad \text{for } (i, j) \in UB. \tag{14}$$

A basis B (more accurately a basis structure (B, LB, UB)) is said to be *dual feasible* if the u_i and v_j determined from (12) satisfy (13) and (14). If we define

$$w_{ij} = \max\,(0, u_i + v_j - c_{ij}) \qquad \text{for } i \epsilon I \text{ and } j \epsilon J, \tag{15}$$

it may be verified that (12)–(15) imply (10) and (11).

By the duality theorem of linear programming [6, 11, 15] a basic solution is optimal if it is both primal and dual feasible. Furthermore, for such a solution

$$Z = \sum_{i \epsilon I} \sum_{j \epsilon J} c_{ij} x_{ij} = \sum_{i \epsilon I} a_i u_i + \sum_{j \epsilon J} b_j v_j - \sum_{i \epsilon I} \sum_{j \epsilon J} U_{ij} w_{ij} = F. \tag{16}$$

EXAMPLE (cont'd): In the tableau of Fig. 1(a) the entry in the northeast corner of each cell gives the value of x_{ij} and is left blank if $x_{ij} \epsilon LB$. c_{ij} and U_{ij} are displayed at the center and the southwest corner of each cell respectively. Denoting the circled cells as the basis B, it may be verified that the solution is basic and primal feasible. The dual variables u_i and v_j appearing at the left and top rims, respectively, satisfy (12). Changing u_i to $u_i + \Delta$ and v_j to $v_j - \Delta$, where Δ is arbitrary, still leaves (12) satisfied. The dual matrix $D = \{d_{ij} = u_i + v_j\}$ of Fig. 1(b), however, is invariant under such a transformation. The reader may verify that the solution is dual feasible (i.e., satisfies (13) and (14) with $LB = \{(1, 2), (2, 4), (3, 2), (3, 3)\}$ and $UB = \{(1, 4), (2, 1)\}$). Setting w_{ij} as given by (15), it could be easily verified that $Z = F = 945$.

3. OPERATORS AND ASSOCIATED SOLUTIONS

We now consider operators that transform the optimum solutions when the data of a problem are changed as a (linear) function of a single parameter.

DEFINITION 5: An *operator* $\delta\mathbf{T}(\mathrm{P})$ transforms the optimum solution of a problem P into that for problem $P^{\mathbf{T}}$ with data:

$$\begin{aligned} a_i^{\mathbf{T}} &= a_i + \delta\alpha_i && \text{for } i \epsilon I, \\ b_j^{\mathbf{T}} &= b_j + \delta\beta_j && \text{for } j \epsilon J, \\ c_{ij}^{\mathbf{T}} &= c_{ij} + \delta\gamma_{ij} && \text{for } i \epsilon I \text{ and } j \epsilon J, \\ U_{ij}^{\mathbf{T}} &= U_{ij} + \delta\upsilon_{ij} && \text{for } i \epsilon I \text{ and } j \epsilon J, \end{aligned} \tag{17}$$

where a_i, b_j, c_{ij} and U_{ij} are the data for problem P, δ is nonnegative, and α_i, β_j, γ_{ij}, and υ_{ij} are given numbers such that $\sum_{i \epsilon I} \alpha_i = \sum_{j \epsilon J} \beta_j$. (This condition is *necessary* so that (A3) is satisfied for the transformed problem $P^{\mathbf{T}}$). We denote this transformation as

(18) $$\delta\mathbf{T}(P, B, LB, UB, X, D, Z) = (P^{\mathrm{T}}, B^{\mathrm{T}}, LB^{\mathrm{T}}, UB^{\mathrm{T}}, X^{\mathrm{T}}, D^{\mathrm{T}}, Z^{\mathrm{T}}),$$

where B, LB, UB, X, D, Z correspond to the optimum solution of problem P and B^{T}, LB^{T}, UB^{T}, X^{T}, D^{T}, and Z^{T} correspond to those for problem P^{T}. In stating (18), we assume the problem P^{T} is also of type P (section 2) and satisfies (A1)–(A3). (It is, of course, possible that P^{T} has no primal feasible solution.)

REMARK 5: In the definition of the operator, we have assumed that $\delta \geqslant 0$. This involves no loss in generality since to study the effects of δ being negative, we can define another operator $\delta'\mathbf{T}(P)$ with $\alpha_i' = -\alpha_i$, $\beta_j' = -\beta_j$, $\gamma_{ij}' = -\gamma_{ij}$, and $v_{ij}' = -v_{ij}$ and still have $\delta' \geqslant 0$.

In most applications an operator as general as that given by Definition 5 is unnecessary. In section 4 we consider *rim operators* ($\delta\mathbf{R}(P)$) which arise when *only* the rim conditions are changed, i.e., $\gamma_{ij} = 0$ and $v_{ij} = 0$ for all $i \epsilon I$ and $j \epsilon J$. *Cost operators* ($\delta\mathbf{C}(P)$) are taken up in section 5 when the changes in the data correspond to cost entries alone, i.e., $\alpha_i = 0$, $\beta_j = 0$ and $v_{ij} = 0$ for all $i \epsilon I$ and $j \epsilon J$. *Bound operators* $\delta\mathbf{L}(P)$ that arise when only the U_{ij}'s are changed (i.e., $\alpha_i = 0$, $\beta_j = 0$, and $\gamma_{ij} = 0$ for $i \epsilon I$ and $j \epsilon J$) are shown in section 6 to be essentially the same as rim operators and hence will not be discussed separately.

TABLE 1. *Operator classification*

Symbol for basis-preserving operator [a]	Name	All the data for the transformed problem are the same as the original problem *except*	Notation for transformed problem [b]	Constraints on δ for basis preserving operator
$\delta T(P)$	operator	See eqn. (17)	$P^T, B^T, LB^T, UB^T, X^T, D^T, Z^T$	$0 \leqslant \delta \leqslant \mu^T$
$\delta R_A(P)$	area rim operator	$a_i^A = a_i + \delta\alpha_i$ for $i \epsilon I$ $b_j^A = b_j + \delta\beta_j$ for $j \epsilon J$ with $\sum_{i \epsilon I} \alpha_i = \sum_{j \epsilon J} \beta_j$	$P^A, B^A, LB^A, UB^A, X^A, D^A, Z^A$	$0 \leqslant \delta \leqslant \mu^A$
$\delta R_{pq}^+(P)$	(plus) cell rim operator	$a_p^+ = a_p + \delta$ $b_q^+ = b_q + \delta$	$P^+, B^+, LB^+, UB^+, X^+, D^+, Z^+$	$0 \leqslant \delta \leqslant \mu^+$
$\delta R_{pq}^-(P)$	(minus) cell rim operator	$a_p^- = a_p - \delta$ $b_q^- = b_q - \delta$	$P^-, B^-, LB^-, UB^-, X^-, D^-, Z^-$	$0 \leqslant \delta \leqslant \mu^-$
$\delta C_A(P)$	area cost operator	$c_{ij}^A = c_{ij} + \delta\gamma_{ij}$ for $i \epsilon I$ and $j \epsilon J$	$P^A, B^A, LB^A, UB^A, X^A, D^A, Z^A$	$0 \leqslant \delta \leqslant \mu^A$
$\delta C_{pq}^+(P)$	(plus) cell cost operator	$c_{pq}^+ = c_{pq} + \delta$	$P^+, B^+, LB^+, UB^+, X^+, D^+, Z^+$	$0 \leqslant \delta \leqslant \mu^+$
$\delta C_{pq}^-(P)$	(minus) cell cost operator	$c_{pq}^- = c_{pq} - \delta$	$P^-, B^-, LB^-, UB^-, X^-, D^-, Z^-$	$0 \leqslant \delta \leqslant \mu^-$

[a] We use bold face letters **T**, **R**, **C** to denote operators (not necessarily basis-preserving).
[b] The notation (P, B, LB, UB, X, D, Z) is used for the original problem.

We call the above operators as *area operators* to distinguish from *cell operators* that arise when a single cost entry, c_{pq}, alone is changed, i.e., $\gamma_{ij}=0$ for $(i, j) \neq (p, q)$ (*cell cost operator*) or when a_p and b_q alone are changed, i.e., $\alpha_i=0$ for $i \neq p$ and $\beta_j=0$ for $j \neq q$ (*cell rim operator*). Thus cell (p, q) refers to the route (p, q) for cost operators and the index pair (p, q) for rim operators. Surprisingly enough many of the applications of operator theory require only cell operators and this is the reason why we prefer to consider it separately in developing efficient algorithms. The cell operators are further classified into + and − operators depending on whether the data are increased or decreased.

DEFINITION 6: An operator $\delta\mathbf{T}(P)$ is said to be *basis preserving*, and denoted by light face letters $\delta T(P)$, if the transformed problem P^T has an optimum solution with $B^T=B$, $LB^T=LB$ and $UB^T=UB$ (i.e., the basis structure is preserved).

We denote by μ^T the maximum value for δ(i.e., $0 \leq \delta \leq \mu^T$) so that the operator is basis preserving. In this paper our discussion is limited to such operators. In [20] we show that any operator can be expressed as a product of basis preserving operators.

Table 1 summarizes the operator classification discussed above.

4. BASIS PRESERVING RIM OPERATORS

In this section we consider rim operators that transform the optimal solutions when only the rim conditions a_i and b_j are changed.

THEOREM 1: For rim operators, the dual solutions are obtained as follows:

(a) For the operator δR^+_{pq}, the solution $D^+=D$, $w^+_{ij}=w_{ij}$ for $i\epsilon I$ and $j\epsilon J$, is dual feasible with $Z^+=Z+\delta d_{pq}$.

(b) For the operator δR^-_{pq}, the solution $D^-=D$, $w^-_{ij}=w_{ij}$ for $i\epsilon I$ and $j\epsilon J$, is dual feasible with $Z^-=Z-\delta d_{pq}$.

(c) For the operator δR_A, the solution $D^A=D$, $w^A_{ij}=w_{ij}$ for $i\epsilon I$ and $j\epsilon J$, is dual feasible with $Z^A=Z+\delta\left[\sum_{i\epsilon I}\alpha_i u_i+\sum_{j\epsilon J}\beta_j v_j\right]$.

PROOF: (a) The duals u_i, v_j, and w_{ij} depend only on the cost entries (which do not change for a rim operator) and the locations of cells in B, LB, and UB (cf. (12) and (15)). Since the operator is basis preserving $B^+=B$. It then follows that $D^+=D$ and $w^+_{ij}=w_{ij}$ for $i\epsilon I$ and $j\epsilon J$. By the basis preserving property $LB^+=LB$ and $UB^+=UB$. Thus the solution is dual feasible (i.e., satisfies (13) and (14)) for problem P^+ since it is for problem P.

At the optimum the values of primal objective function Z^+ and the dual objective function F^+ are the same so that from (16)

$$Z^+-Z=F^+-F=\sum_{i\epsilon I}(a^+_i-a_i)\,u_i+\sum_{j\epsilon J}(b^+_j-b_j)\,v_j-\sum_{i\epsilon I}\sum_{j\epsilon J}(U^+_{ij}-U_{ij})\,w_{ij}.$$

By the definition of δR^+_{pq} (Table 1), $a^+_i=a_i$ for $i\neq p$ with $a^+_p=a_p+\delta$, $b^+_j=b_j$ for $j\neq q$ with $b^+_q=b_q+\delta$ and $U^+_{ij}=U_{ij}$ for $i\epsilon I$ and $j\epsilon J$ so that

$$Z^+-Z=\delta u_p+\delta v_q=\delta(u_p+v_q)=\delta d_{pq}.$$

The proofs for (b) and (c) are similar.

REMARK 6: In section 2 we remarked that u_i and v_j are determined only up to an additive constant, i.e., $u_i' = u_i + \Delta$ for $i \epsilon I$ and $v_j' = v_j - \Delta$ for $j \epsilon J$, also satisfy (12). However, the cost effect given by Theorem 1(c) viz., $\delta\left[\sum_{i \epsilon I} \alpha_i u_i + \sum_{j \epsilon J} \beta_j v_j\right]$ is invariant under such a transformation since we have assumed that the parameters α_i and β_j satisfy the condition $\sum_{i \epsilon I} \alpha_i = \sum_{j \epsilon J} \beta_j$.

The next three theorems give the primal solution to the problems obtained after applying rim operators of various kinds.

THEOREM 2:

(a) For the operator δR_{pq}^+ with $(p, q) \epsilon B$, the optimal primal solution X^+ to the problem P^+ is given by

$$x_{ij}^+ = x_{ij} \qquad \text{for } (i, j) \epsilon [(I \times J) - \{(p, q)\}],\ x_{pq}^+ = x_{pq} + \delta, \tag{19}$$

where $\times$ denotes cross product of sets. Furthermore the maximum extent to which this basis preserving operator can be applied is given by

$$\mu^+ = (U_{pq} - x_{pq}). \tag{20}$$

(b) For the operator δR_{pq}^- with $(p, q) \epsilon B$,

$$x_{ij}^- = x_{ij} \qquad \text{for } (i, j) \epsilon [(I \times J) - \{(p, q)\}], \qquad x_{pq}^- = x_{pq} - \delta, \tag{21}$$

$$\mu^- = x_{pq}. \tag{22}$$

PROOF: It can be easily verified that X^+ given by (19) is a basic solution (i.e., satisfies (4) and (5) for problem P^+) and provided $\delta \leq \mu^+$ the solution is primal feasible (i.e., satisfies (6)). By Theorem 1(a) the solution is dual feasible as well and hence optimal. The quantity μ^+ gives the maximum permissible δ for a basis preserving operator since for $\delta > \mu^+$, x_{pq}^+ violates (6). Similar comments apply to (b).

THEOREM 3:

(a) For the operator δR_{pq}^+ with $(p, q) \notin B$,

$$x_{ij}^+ = \begin{cases} x_{ij} + \delta & \text{for } (i, j) \epsilon \Gamma_1 \\ x_{ij} - \delta & \text{for } (i, j) \epsilon [\Gamma_2 - \{(p, q)\}] \\ x_{ij} & \text{elsewhere,} \end{cases} \tag{23}$$

where Γ is the cycle obtained by adding (p, q) to B, Γ_1 and Γ_2 are the cells in Γ with odd and even parity (Definition 3). The maximum extent to which this basis preserving operator can be applied is given by

$$\mu^+ = \text{Min} \begin{cases} (U_{ij} - x_{ij}) & \text{for } (i, j) \epsilon \Gamma_1 \\ x_{ij} & \text{for } (i, j) \epsilon [\Gamma_2 - \{(p, q)\}]. \end{cases} \tag{24}$$

(b) For the operator δR^-_{pq} with $(p, q) \notin B$,

$$(25) \qquad x^-_{ij} = \begin{cases} x_{ij} - \delta & \text{for } (i,j) \in \Gamma_1 \\ x_{ij} + \delta & \text{for } (i,j) \in [\Gamma_2 - \{(p,q)\}] \\ x_{ij} & \text{elsewhere,} \end{cases}$$

$$(26) \qquad \mu^- = \mathrm{Min} \begin{cases} x_{ij} & \text{for } (i,j) \in \Gamma_1 \\ (U_{ij} - x_{ij}) & \text{for } (i,j) \in [\Gamma_2 - \{(p,q)\}], \end{cases}$$

where Γ_1 and Γ_2 are as defined in (a).

PROOF: (a) We first show that X^+ as given by (23) is a basic solution, i.e., it satisfies (4) and (5) for the problem P^+. From Remark 4 row p has $(p, q) \in \Gamma_2$ and a cell, say (p, h), belonging to Γ_1. From (23) $x^+_{pq} = x_{pq}$ and $x^+_{ph} = x_{ph} + \delta$. Since there are no other cells of Γ in line p, $\sum_{j\epsilon J} x^+_{pj} = \delta + \sum_{j\epsilon J} x_{pj}$. For other rows $i \neq p$, from (23) and Remark 4 it follows that $\sum_{j\epsilon J} x^+_{ij} = \sum_{j\epsilon J} x_{ij}$. Since X is a basic solution to P it satisfies (4) viz. $\sum_{j\epsilon J} x_{ij} = a_i$. Thus $\sum_{j\epsilon J} x^+_{ij} = a_i$ for $i \neq p$ and $\sum_{j\epsilon J} x^+_{pj} = a_p + \delta = a^+_p$. Similarly $\sum_{i\epsilon I} x^+_{ij} = b_j$ for $j \neq q$ and $\sum_{i\epsilon I} x^+_{iq} = b_q + \delta = b^+_q$. It then easily follows that X^+ is a basic solution. Also by the choice of μ^+ in (24), the solution X^+ for $\delta \leq \mu^+$ can be verified to be primal feasible. By Theorem 1(a) the solution is dual feasible as well and hence optimal. The quantity μ^+ gives the maximum δ for basis preserving operator since for $\delta > \mu^+$, x^+_{rs} corresponding to the *leaving cell** (r, s) (i.e., the cell at which (24) attains the minimum) can be shown to violate (6).

The proof for part (b) is similar.

To consider (basis preserving) area rim operator, we determine a *modification matrix* Y such that, when x_{ij} is replaced by $x_{ij} + \delta y_{ij}$, the conditions (4) and (5) are satisfied for the problem P^A.

DEFINITION 7: Given a basis B, the modification matrix $Y = \{y_{ij}\}$ associated with the coefficients α_i for $i\epsilon I$ and β_j for $j \epsilon J$, is such that $y_{ij} = 0$ for $(i, j) \notin B$, and

$$(27) \qquad \sum_{j\epsilon J} y_{ij} = \alpha_i \qquad \text{for } i\epsilon I, \text{ and}$$

$$(28) \qquad \sum_{i\epsilon I} y_{ij} = \beta_j \qquad \text{for } j\epsilon J.$$

REMARK 7: Recalling that $\sum_{i\epsilon I} \alpha_i = \sum_{j\epsilon J} \beta_j$, the determination of y_{ij} from (27)–(28) will be seen to be similar to the determination of x_{ij} from (4)–(5). However, α_i, β_j, and y_{ij} unlike a_i, b_j, and x_{ij} can be negative. Though x_{ij} can be nonzero for the nonbasic cells, this is not allowed for the entries y_{ij}. To determine the amounts y_{ij} corresponding to the basic cells, we can use the following "crossing-out routine"

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*To apply the operator δ**R** with $\delta > \mu$, we show in [20] that the cell $\{(r, s)\}$ *leaves* the basis and some cell $\{(e, f)\}$ enters the basis so that $B - \{(r, s)\} + \{(e, f)\}$ is an adjacent dual feasible basis.

[11]. In effect, we are solving for the unique y_{ij}'s by using the triangularity property of the basis [6, 8, 11, 15]. At the start, none of the lines (rows or columns) or basic cells is crossed out.

(A) Choose an uncrossed line such that it has exactly one cell not yet crossed out and proceed to (B). If there is no such line, STOP since all the y_{ij}'s corresponding to basic cells, satisfying (27) and (28), have been determined.

(B) If the line is row i and the unique cell is (i, j) set $y_{ij}=\alpha_i$, modify β_j to β_j-y_{ij}, cross out row i and the cell (i, j). Go to (A).

If the line is column j and the unique cell is (i, j), set $y_{ij}=\beta_j$, modify α_i to α_i-y_{ij}, cross out column j and the cell (i, j). Go to (A).

THEOREM 4: For the operator δR_A, the optimum solution X^A is given by

$$x^A_{ij}=x_{ij}+\delta y_{ij} \qquad \text{for } i\epsilon I \text{ and } j\epsilon J. \tag{29}$$

The maximum extent to which this basis preserving operator can be applied is given by

$$\mu^A=\text{Min}\begin{cases}(U_{ij}-x_{ij})/y_{ij} & \text{for } \{(i, j)\ \epsilon\ B|y_{ij}>0\}\\ -x_{ij}/y_{ij} & \text{for } \{(i, j)\ \epsilon\ B|y_{ij}<0\}.\end{cases} \tag{30}$$

PROOF: We first show that the solution X^A defined in (29) is basic, i.e., satisfies (4) and (5) for P^A. Since $y_{ij}=0$ for $(i, j) \notin B$, $x^A_{ij}=x_{ij}$ for these cells. From (29) $\sum_{j\epsilon J} x^A_{ij}=\sum_{j\epsilon J} x_{ij}+\delta\sum_{j\epsilon J} y_{ij}$. Since X is a basic solution for P, $\sum_{j\epsilon J} x_{ij}=a_i$. From (27), $\sum_{j\epsilon J} y_{ij}=\alpha_i$ so that $\sum_{j\epsilon J} x^A_{ij}=a_i+\delta\alpha_i=a^A_i$ for $i\epsilon I$ so that (4) and similarly (5) are satisfied for the problem T^A. By the choice of μ^A in Equation (30), the solution X^A is primal feasible for all $0\leqslant\delta\leqslant\mu^A$. By Theorem 1(c) the solution is dual feasible as well and hence optimal.

The quantity μ^A gives the maximum δ for δR_A to be basis preserving since for $\delta>\mu^A$, x^A_{rs} corresponding to the *leaving cell* (r, s) (i.e., the cell at which the minimum of (30) is taken on) can be shown to violate (6).

We summarize the results of this section in the form of Algorithm 1 for determining μ and Algorithm 2 for making the transformation of optimal solution associated with a basis preserving rim operator.

ALGORITHM 1: For determining μ associated with a basis preserving rim operator and the leaving cell (r, s).

(1) If μ^A is to be determined (δR_A) go to (4). If $\mu^\pm$ is to be determined $(\delta R^\pm_{pq})$ go to (2).

(2) If $(p, q)\notin B$ go to (3). Otherwise determine $\mu^+(\mu^-)$ from (20) ((22)). Record $(r, s)=(p, q)$. STOP.

(3) Find the cycle Γ obtained when (p, q) is added to B. Using Definition 3 identify Γ_1 and Γ_2. Determine $\mu^+(\mu^-)$ from (24) ((26)). Record (r, s) as the cell at which the minimum of (24) ((26)) is taken on. STOP.

(4) Find $Y=\{y_{ij}\}$ using Remark 7. Determine μ^A from (30). Record (r, s) as the cell at which the minimum of (30) is taken on. STOP.

ALGORITHM 2: For applying the basis preserving rim operator.

(1) If it is an area operator (δR_A) go to (4). If it is a cell operator ($\delta R_{pq}^{\pm}$) go to (2).

(2) Set the data of $P^{\pm}$ same as P except that $a_p^{\pm}=a_p\pm\delta$, $b_q^{\pm}=b_q\pm\delta$. Set $B^{\pm}=B$, $LB^{\pm}=LB$, $UB^{\pm}=UB$, $D^{\pm}=D$ and $Z^{\pm}=Z\pm\delta d_{pq}$. If $(p, q)\notin B$ go to (3). Otherwise determine $X^+(X^-)$ using (19) ((21)). STOP.

(3) Find the cycle Γ obtained by adding (p, q) to the basis. Using Definition 3 identify Γ_1 and Γ_2. Determine X^+ (X^-) using (23) ((25)). STOP.

(4) Set the data of problem P^A same as P except that $a_i^A=a_i+\delta\alpha_i$ for $i\epsilon I$, $b_j^A=b_j+\delta\beta_j$ for $j\epsilon J$. Set $B^A=B$, $LB^A=LB$, $UB^A=UB$, $D^A=D$ and $Z^A=Z+\delta\left[\sum_{i\epsilon I}\alpha_iu_i+\sum_{j\epsilon J}\beta_jv_j\right]$. Find $Y=\{y_{ij}\}$ using Remark 7. Find X^A using (29). STOP.

EXAMPLE (cont'd): Referring to Fig. 1(a), for the operator δR_{11}^+, we find $\mu^+=N-25$ from (20). Since N is a large positive number this implies that this operator can be applied to any large extent. On the other hand, the operator $\delta R_{\bar{2}\bar{3}}$ can be applied to a maximum extent of $\mu^-=10$ from (22). For instance, if $\delta=5$, the optimum solution will have $x_{\bar{2}\bar{3}}=5$ with all other x_{ij}'s remaining the same (21). It can be easily verified that the solution is primal feasible with $a_{\bar{2}}=85$ and $b_{\bar{3}}=30$. The solution is dual feasible and hence optimal. The optimal cost $Z^-=Z-\delta\times d_{23}=945-5\times 8=905$ which is true since the only change in optimal cost is due to the decrease of x_{23} at the rate of $c_{23}=d_{23}=8$.

From Fig. 1(b) we see that $d_{32}=-2$ which means the total cost will *decrease* at the rate of 2 if a_3 and b_2 were simultaneously *increased*. This is an instance of "transportation paradox" [3] or shipping more for less. We apply the operator δR_{32}^+ using Algorithm 2. In step (3) of the algorithm, we determine the cycle shown in Fig. 1(c) with $\Gamma_1=\{(3, 1), (1, 3), (2, 2)\}$ and $\Gamma_2=\{(3, 2), (1, 1), (2, 3)\}$. Using (23) we get the solution shown in Fig. 1(c) for general δ and Fig. 1(d) for $\delta=5$. Cost increases at the rate $c_{31}+c_{13}+c_{22}=1+3+5=9$ (on the cells of odd parity), but decreases at the rate $c_{11}+c_{23}=3+8=11$ (on the cells of even parity) so that the net *decrease* is at the rate of 2. We find $\mu^+=5$ limited by the cell (1, 3) reaching its upper bound.

We now consider an area operator δR_A for which the α_i and β_j are shown on the rims of Fig. 1(e). The values of $Y=\{y_{ij}\}$ determined using Remark 7 are shown at the northeast corner of the basis cells of Fig. 1(e) ($y_{ij}=0$ for the nonbasic cells). From (30) and using the x_{ij} values of Fig. 1(a) we get $\mu^A=2$, this time being limited by cell (2, 3). Application of the operator $2R_A$ results in the optimal solution of Fig. 1(f). To find the cost effect of this operator we compute $\sum_{i\epsilon I}\alpha_iu_i+\sum_{j\epsilon J}\beta_jv_j=-14$ so that from Theorem 1(c), $Z^A=Z-2\times 14=945-28=917$ which checks with the total cost $\sum_{i\epsilon I}\sum_{j\epsilon J}c_{ij}x_{ij}^A$ computed from Fig. 1(f).

In this problem all the μ's turn out to be strictly positive. It is easy to infer from (20), (22), (24), (26), and (30) that this is not necessarily the case if the solution is (*primal*) *degenerate*, i.e., x_{ij} corresponding to some basic cell (i, j) takes on either its lower or upper bound.

5. BASIS-PRESERVING COST OPERATORS

We now consider cost operators that transform the optimal solutions when only the cost elements, c_{ij}, are changed in such a way that the optimal basis is preserved.

THEOREM 5: For cost operators the primal solutions are determined as follows:

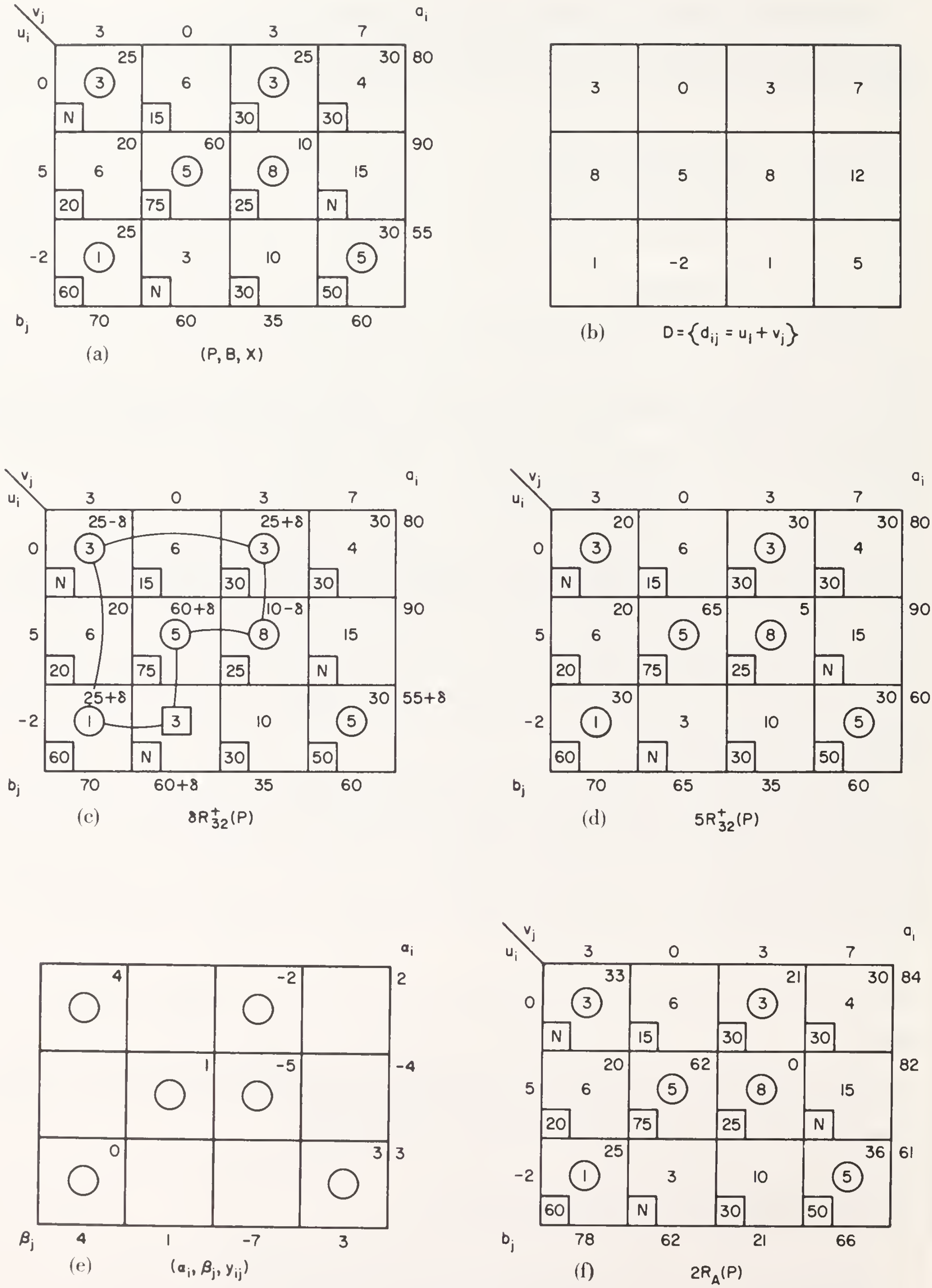

FIGURE 1.

(a) For the operator δC_{pq}^{+}, $X^{+}=X$ and $Z^{+}=Z+\delta x_{pq}$.

(b) For the operator δC_{pq}^{-}, $X^{-}=X$ and $Z^{-}=Z-\delta x_{pq}$.

(c) For the operator δC_{A}, $X^{A}=X$ and $Z^{A}=Z+\delta\left[\sum_{i\epsilon I}\sum_{j\epsilon J}\gamma_{ij}x_{ij}\right]$.

PROOF: (a) Since the operator is basis preserving, $LB^{+}=LB$ and $UB^{+}=UB$ so that $x_{ij}^{+}=x_{ij}$ for these nonbasic cells. Since it is a cost operator, $a_i^{+}=a_i$ for $i\epsilon I$ and $b_j^{+}=b_j$ for $j\epsilon J$. It then follows that $x_{ij}^{+}=x_{ij}$ for $(i, j)\epsilon B=B^{+}$. Thus $X^{+}=X$. $Z^{+}=\sum_{i\epsilon I}\sum_{j\epsilon J}c_{ij}^{+}x_{ij}^{+}=\sum_{i\epsilon I}\sum_{j\epsilon J}c_{ij}^{+}x_{ij}$. By definition of δC_{pq}^{+}, $c_{ij}^{+}=c_{ij}$ for $(i, j)\neq(p, q)$, $c_{pq}^{+}=c_{pq}+\delta$, so that $Z^{+}=\sum_{i\epsilon I}\sum_{j\epsilon J}c_{ij}x_{ij}+\delta x_{pq}=Z+\delta x_{pq}$.

The proofs for parts (b) and (c) are similar.

Theorems 6–8 determine the optimal dual solutions to the problems obtained after applying the cost operators of various kinds. We first consider cell operators with $(p, q)\notin B$.

THEOREM 6:

(a) For the operator δC_{pq}^{+} with $(p, q)\notin B$, $D^{+}=D$ and

$$\mu^{+}=\begin{cases} M & \text{if } (p, q)\ \epsilon\ LB \\ u_p+v_q-c_{pq} & \text{if } (p, q)\ \epsilon\ UB. \end{cases} \tag{31}$$

(b) For the operator δC_{pq}^{-} with $(p, q)\notin B$, $D^{-}=D$ and

$$\mu^{-}=\begin{cases} c_{pq}-u_p-v_q & \text{if } (p, q)\ \epsilon\ LB \\ M & \text{if } (p, q)\ \epsilon\ UB, \end{cases} \tag{32}$$

where M is an arbitrarily large positive number.

PROOF:

(a) Since $B^{+}=B$ and $c_{ij}^{+}=c_{ij}$ for $(i, j)\ \epsilon\ B$, from (12) it follows that $u_i^{+}=u_i$ for $i\epsilon I$ and $v_j^{+}=v_j$ for $j\epsilon J$ so that $D^{+}=D$. For every $(i, j)\notin B$ with $(i, j)\neq(p, q)$, $c_{ij}^{+}=c_{ij}$ and the duals u_i and v_j are unchanged so that the dual feasibility condition (13) or (14) is satisfied, since it is for problem P. For the cell (p, q) itself, two cases arise:

(i) $(p, q)\ \epsilon\ LB$; $c_{pq}^{+}-u_p^{+}-v_q^{+}=c_{pq}+\delta-u_p-v_q\geqslant 0$ since $\delta\geqslant 0$ and $c_{pq}-u_p-v_q\geqslant 0$ since the solution is dual feasible for P. Since there is no upper bound for δ we impose an arbitrary bound of M.

(ii) $(p, q)\ \epsilon\ UB$: $c_{pq}^{+}-u_p^{+}-v_q^{+}=c_{pq}+\delta-u_p-v_q\leqslant 0$ provided $\delta\leqslant u_p+v_q-c_{pq}=\mu^{+}$. For $\delta>\mu^{+}$, the cell (p, q) would lose dual feasibility.

Thus for $0\leqslant\delta\leqslant\mu^{+}$, the solution $D^{+}=D$ is dual feasible. Since $X^{+}=X$ is primal feasible (rims have not changed), the solution is optimal as well. The proof for (b) is similar.

For operators δC_{pq}^{+} with $(p, q)\ \epsilon\ B$, to study the changes in u_i and v_j associated with a change in c_{pq} we introduce the following definition.

DEFINITION 8: Let $\Omega=B-\{(p, q)\}$ where $(p, q)\ \epsilon\ B$. Then the sets I_p and I_q defined below *partition* I, i.e., $I_p\cup I_q=I$ and $I_p\cap I_q=\phi$. Similarly J_p and J_q partition J.

$$
\begin{aligned}
I_p &= \{p\} \cup \{i \in I \mid i \text{ is connected to } p \text{ in } \Omega\} \\
J_p &= \{j \in J \mid j \text{ is connected to } p \text{ in } \Omega\} \\
I_q &= \{i \in I \mid i \text{ is connected to } q \text{ in } \Omega\} \\
J_q &= \{q\} \cup \{j \in J \mid j \text{ is connected to } q \text{ in } \Omega\}.
\end{aligned} \tag{33}
$$

To justify that I_p and I_q partition I, we note from Definition 2 that row p is connected to every other row $h \in I - \{p\}$ in B by an *unique* path (7) $\{(i_1, j_1), (i_2, j_2), \ldots (i_k, j_k)\}$ in which (i_1, j_1) is the only cell in row p. Two cases arise corresponding to whether this path requires the cell (p, q) or not.

(a) $(i_1, j_1) \neq (p, q)$. Then all the elements of the path (7) belong to $\Omega = B - \{(p, q)\}$. Thus $h \in I_p$.

(b) $(i_1, j_1) = (p, q)$. Since the path (7) is unique and $(p, q) \notin \Omega$, row p is *not* connected to h in Ω. Thus $h \notin I_p$. But the sequence $S' = S - \{(i_1, j_1)\}$ connects column q to row h in Ω (By property (b) in Definition 1, (i_2, j_2) is a cell in line q). Thus $h \in I_q$.

It then follows that $I_p \cup I_q = I$ and $I_p \cap I_q = \phi$. Similar arguments apply to J_p and J_q. The partitioning of Definition 8 is also useful in finding an adjacent dual feasible basis [1, 2, 12, 20] and in the time-minimizing transportation problem [21].

REMARK 8: If (p, q) is the unique cell in row p of basis B then $I_p = \{p\}$, and $J_p = \phi$. If (p, q) is unique in column q then $I_q = \phi$ and $J_q = \{q\}$. Since $(p, q) \in B$ it cannot be unique in both row p and column q. Thus at most one of the sets I_q and J_p can be empty.

REMARK 9: The following "*scanning routine*" can be used for identifying I_p, J_p, I_q, and J_q. The sets LI_p and LJ_p used in the routine below may be interpreted as the "labeled, but not yet scanned" subsets of I_p and J_p [17].

(i) Set $I_p = LI_p = \{p\}$; Set $J_p = \phi$.

(ii) If $LI_p = \phi$ go to (iv). Otherwise set $LJ_p = \phi$. Carryout (a) below for every $i \in LI_p$ and proceed to (iii).

(a) $Q_i = \{j \in J - J_p \mid (i, j) \in B - (p, q)\}$. $J_p = J_p \cup Q_i$, $LJ_p \cup Q_i$.

(iii) If $LJ_p = \phi$ go to (iv). Otherwise set $LI_p = \phi$. Carryout (b) below for every $j \in LJ_p$ and proceed to (ii).

(b) $R_j = \{i \in I - I_p \mid (i, j) \in B - (p, q)\}$. $I_p = I_p \cup R_j$, $LI_p = LI_p \cup R_j$.

(iv) Set $I_q = I - I_p$ and $J_q = J - J_p$. STOP.

That this algorithm identifies the subsets I_p, J_p, I_q, and J_q follows from Definitions 1 and 8.

EXAMPLE (cont'd): The reader may verify using the above algorithm for the basis of Fig. 1(a), that the subsets corresponding to $(p, q) = (1, 1)$ are given by $I_p = \{1, 2\}$, $J_p = \{2, 3\}$, $I_q = \{3\}$, and $J_q = \{1, 4\}$. The partition corresponding to $(p, q) = \{(2, 2)\}$ is $I_p = I = \{1, 2, 3\}$, $J_p = \{1, 3, 4\}$, $I_q = \phi$, and $J_q = \{2\}$ verifying Remark 8 for the case when (p, q) is unique in its column.

REMARK 10: By the construction of the sets I_p, I_q, J_p, and J_q (Definition 8 and Remark 9) $B - \{(p, q)\} \subset [(I_p \times J_p) \cup (I_q \times J_q)]$ with $(p, q) \in [I_p \times J_q]$.

THEOREM 7:

(a) For δC_{pq}^{+} with $(p, q) \in B$, the optimal dual solutions are given by

$$
u_i^{+} = \begin{cases} u_i + \delta & \text{for } i \in I_p \\ u_i & \text{for } i \in I_q, \end{cases}
$$

(34)
$$v_j^+ = \begin{cases} v_j - \delta & \text{for } j \in J_p \\ v_j & \text{for } j \in J_q. \end{cases}$$

The maximum extent μ^+ to which this basis preserving operator can be applied is given by

(35)
$$\mu^+ = \text{Min} \begin{cases} (c_{ij} - u_i - v_j) & \text{for } (i, j) \in [(I_p \times J_q) \cap LB] \\ (u_i + v_j - c_{ij}) & \text{for } (i, j) \in [(I_q \times J_p) \cap UB]. \end{cases}$$

(b) For δC_{pq}^- with $(p, q) \in B$, the optimal dual solutions are given by

(36)
$$u_i^- = \begin{cases} u_i - \delta & \text{for } i \in I_p \\ u_i & \text{for } i \in I_q, \end{cases}$$

$$v_j^- = \begin{cases} v_j + \delta & \text{for } j \in J_p \\ v_j & \text{for } j \in J_q. \end{cases}$$

Furthermore

(37)
$$\mu^- = \text{Min} \begin{cases} (u_i + v_j - c_{ij}) & \text{for } (i, j) \in [(I_p \times J_q) \cap UB] \\ (c_{ij} - u_i - v_j) & \text{for } (i, j) \in [(I_q \times J_p) \cap LB]. \end{cases}$$

The minimum in (35) and (37) is set equal to M if the relevant sets over which the minimization is done is empty.

PROOF: (a) We note that $u_i^+ + v_j^+ = u_i + v_j$ for $(i, j) \in (I_p \times J_p) \cup (I_q \times J_q)$. By Remark 10 all the basic cells except (p, q) are contained in $(I_p \times J_p) \cup (I_q \times J_q)$. Since $c_{ij}^+ = c_{ij}$ for $(i, j) \neq (p, q)$, (12) is satisfied by these basis cells for the problem P^+. For the basis cell (p, q), $c_{pq}^+ = c_{pq} + \delta = u_p + v_q + \delta = u_p^+ + v_q^+$. Thus the solutions (34) satisfy (12) for problem P^+.

For the nonbasic cells $(i, j) \in (I_p \times J_p) \cup (I_q \times J_q)$, $c_{ij}^+ = c_{ij}$ and $u_i^+ + v_j^+ = u_i + v_j$ so that the dual feasibility conditions (13) and (14) are satisfied. For $(i, j) \in I_p \times J_q$, $u_i^+ + v_j^+ = u_i + v_j + \delta$ and for $(i, j) \in I_q \times J_p$, $u_i^+ + v_j^+ = u_i + v_j - \delta$ so that by the choice of μ^+ in (35) the dual feasibility conditions (13) and (14) are satisfied in these areas as well for $0 \leq \delta \leq \mu^+$. Since the solution $X^+ = X$ is primal feasible as well, it is optimal. μ^+ gives the maximum value for δ, since for $\delta > \mu^+$ dual feasibility condition will be violated by the *entering cell** (e, f) (at which the minimum of (35) is taken on).

The proof for part (b) is similar.

REMARK 11: Let us assume that for the basic cell (p, q), $0 < x_{pq} < U_{pq}$. If $\mu^+ = M$ this means that regardless of how large the cost c_{pq} may be, the optimal solution continues to ship a positive amount along this route. In other words there is no primal feasible solution to the problem with $x_{pq} = 0$.

---

*To apply the operator δC with $\delta > \mu$, we show in [20] that the cell $\{(e, f)\}$ *enters* the basis and some cell $\{(r, s)\}$ leaves the basis so that $B + \{(e, f)\} - \{(r, s)\}$ is an adjacent primal feasible basis. Alternatively the cell (e, f) changes from LB to UB or vice versa.

Similarly we would expect x_{pq} to increase when c_{pq} is decreased. Thus the case $\mu^- = M$ would mean that x_{pq} has already reached its maximum value ($x_{pq} = \min(a_p, b_q, U_{pq})$). Both these results can be proved more rigorously by studying the structure of the cells in B, LB, and UB for the cases when the sets over which the minimization is done turn out to be empty.

We now take up the case of area cost operator δC_A for which the cost entry c_{ij} becomes $c_{ij} + \delta\gamma_{ij}$ for $i \epsilon I$ and $j \epsilon J$. To study the effect of the cost changes on the dual variables, we define u_i^* for $i \epsilon I$ and v_j^* for $j \epsilon J$ such that

$$u_i^* + v_j^* = \gamma_{ij} \qquad \text{for } (i, j) \epsilon B, \tag{38}$$

i.e., u_i^*, v_j^* are determined from γ_{ij} in the same manner as u_i and v_j are computed from c_{ij}.

THEOREM 8: For the cost operator δC_A, the optimal dual solutions are given by

$$\begin{aligned} u_i^A &= u_i + \delta u_i^* && \text{for } i \epsilon I \\ v_j^A &= v_j + \delta v_j^* && \text{for } j \epsilon J. \end{aligned} \tag{39}$$

The maximum extent to which this basis preserving operator can be applied is given by

$$\mu^A = \operatorname{Min} \begin{cases} (c_{ij} - u_i - v_j)/(u_i^* + v_j^* - \gamma_{ij}) & \text{for } (i, j) \epsilon LB \mid (\gamma_{ij} - u_i^* - v_j^*) < 0 \\ (u_i + v_j - c_{ij})/(\gamma_{ij} - u_i^* - v_j^*) & \text{for } (i, j) \epsilon UB \mid (\gamma_{ij} - u_i^* - v_j^*) > 0 \end{cases} \tag{40}$$

μ^A is set equal to M, if the sets over which the minimization is done turn out to be empty.

PROOF: First we show that the dual solutions (39) satisfy (12) for the problem P^A. From (38) and (39) we get $u_i^A + v_j^A = (u_i + v_j) + \delta(u_i^* + v_j^*) = c_{ij} + \delta\gamma_{ij} = c_{ij}^A$. For $(i, j) \epsilon LB$, $c_{ij}^A - u_i^A - v_j^A = (c_{ij} - u_i - v_j) + \delta(\gamma_{ij} - u_i^* - v_j^*)$. Since $c_{ij} - u_i - v_j \geqslant 0$ for the problem P, for $0 \leqslant \delta \leqslant \mu^A$ as defined by (40) the dual feasibility condition (13) is satisfied by these cells. For $(i, j) \epsilon UB$, $u_i^A + v_j^A - c_{ij}^A = (u_i + v_j - c_{ij}) + \delta(u_i^* + v_j^* - \gamma_{ij})$. Since $u_i + v_j - c_{ij} \geqslant 0$ for the problem P, for $0 \leqslant \delta \leqslant \mu^A$ as defined by (40) the dual feasibility condition (14) is satisfied by these cells.

Thus the solution (34) is dual feasible. Since $X^A = X$ is primal feasible as well, it is optimal. The quantity μ^A gives the maximum δ, since for $\delta > \mu^A$, the *entering cell* (e, f) (at which the minimum of (40) is taken on) will violate dual feasibility.

We summarize the results of this section in the form of Algorithm 3 for determining μ and Algorithm 4 for applying the operator δC.

ALGORITHM 3: For determining μ associated with a basis-preserving cost operator and the entering cell (e, f).

(1) If μ^A is to be determined (δC_A) go to (4). If $\mu^{\pm}$ is to be determined ($\delta C_{pq}^{\pm}$) go to (2).

(2) If $(p, q) \epsilon B$ go to (3). Otherwise determine μ^+ (μ^-) using (31) ((32)). Record $(e, f) = (p, q)$. STOP.

(3) Find the sets I_p, J_p, I_q, and J_q using the scanning routine of Remark 9. Determine μ^+ (μ^-) from (35) ((37)). Record (e, f) as the cell at which the minimum is taken on. STOP.

(4) Find u_i^*, v_j^* satisfying (38). Determine μ^A using (40). Record (e, f) as the cell at which the minimum is taken on. STOP.

ALGORITHM 4: For applying the basis-preserving cost operator.

(1) If it is an area operator (δC_A) go to (4). If it is a cell operator ($\delta C^{\pm}_{pq}$) go to (2).

(2) Set the data of problem $P^{\pm}$ same as problem P except that $c^{\pm}_{pq} = c_{pq} \pm \delta$. Set $B^{\pm} = B$, $LB^{\pm} = LB$, $UB^{\pm} = UB$, $X^{\pm} = X$, and $Z^{\pm} = Z \pm \delta x_{pq}$. If $(p, q) \in B$ go to (3). Otherwise set $D^{\pm} = D$. STOP.

(3) Find the sets I_p, J_p, I_q, and J_q using the scanning routine of Remark 9. Determine D^+ (D^-) using (34) ((36)). STOP.

(4) Set the data of P^A same as P except that $c^A_{ij} = c_{ij} + \delta\gamma_{ij}$ for $i \in I$ and $j \in J$. Set $B^A = B$, $LB^A = LB$, $UB^A = UB$, $X^A = X$, and $Z^A = Z + \delta\left[\sum_{i \in I} \sum_{j \in J} \gamma_{ij} x_{ij}\right]$. Find u^*_i and v^*_j satisfying (38). Determine D^A using (39). STOP.

EXAMPLE (cont'd): Referring to Fig. 1(a), for the operator δC^+_{12}, from (31) we get $\mu^+ = M$; i.e., the solution of Fig. 1(a) is optimal regardless of how large c_{12} becomes. For the operator δC^-_{12}, however, $\mu^- = 6$ from (32). Similarly $\mu^+ = 3$ for δC^+_{14}. For all these operators the optimum solutions X and D remain the same.

Consider now the operator δC^+_{11}. Using the scanning routine of Remark 9, we get $I_p = \{1, 2\}$, $J_p = \{2, 3\}$, $I_q = \{3\}$, $J_q = \{1, 4\}$. Using (34) we get the dual solutions of Fig. 2(a) for general δ and Fig. 2(b) for $\delta = 3$. Cost increases at the rate of $x_{11} = 25$ since none of the other costs are affected (Theorem 5(a)). We find $\mu^+ = 3$ limited by the cell $(e, f) = (2, 4)$. The reader may verify that the dual solutions of Fig. 2(a) and 2(b) are dual feasible (i.e., satisfy (13) and (14)). Since the primal solution remains the same it is optimal as well.

We now consider δC_A with γ_{ij} as shown in Fig. 2(c). The values of u^*_i and v^*_j satisfying (38) are displayed on the left and top rims of Fig. 2(c). Using (40), $\mu_A = 1$ and the minimum is attained at the cell $(e, f) = (1, 2)$. Application of $1C_A$ using Algorithm 4 results in the tableau of Fig. 2(d). Again the solution may be verified to be dual feasible and hence optimal. From Theorem 5(c), $Z^A = 945 + 860 = 1805$ which agrees with $\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$ directly computed from Fig. 2(d).

In this problem all the μ's turn out to be strictly positive. From (31), (32), (35), (37), and (40) it is easy to see that this need not be the case if the solution is dual degenerate (i.e., corresponding to a nonbasic cell (i, j), $c_{ij} - u_i - v_j = 0$).

6. GENERALIZATIONS OF OPERATOR THEORY

In this section we first show that bound operators, that arise when the upper bounds alone are changed, can be reduced to an application of rim operators. We also show that simultaneous changes of costs and rim conditions can be effected by sequentially applying rim and cost operators. As usual, our discussion will be limited to basis preserving operators.

DEFINITION 9: A *bound operator* $\delta L(P)$ transforms the optimum solutions of a problem P into those of a transformed problem whose data are the same as those of P except that

(a) $U^+_{pq} = U_{pq} + \delta$ for the δL^+_{pq} operator where (p, q) is a given index pair,

(b) $U^-_{pq} = U_{pq} - \delta$ for the δL^-_{pq} operator where (p, q) is a given index pair, and

(c) $U^A_{ij} = U_{ij} + \delta v_{ij}$ for $i \in I$ and $j \in J$ for the δL_A operator.

For $\delta L(P)$ to be basis-preserving, by Definition 6 we have $B^{\pm}(B^A) = B$, $LB^{\pm}(LB^A) = LB$ and $UB^{\pm}(UB^A) = UB$. The last of these equations has to be interpreted as: the cells that are nonbasic at their upper bounds for P still remain nonbasic at their (possibly different) upper bounds for the transformed problem.

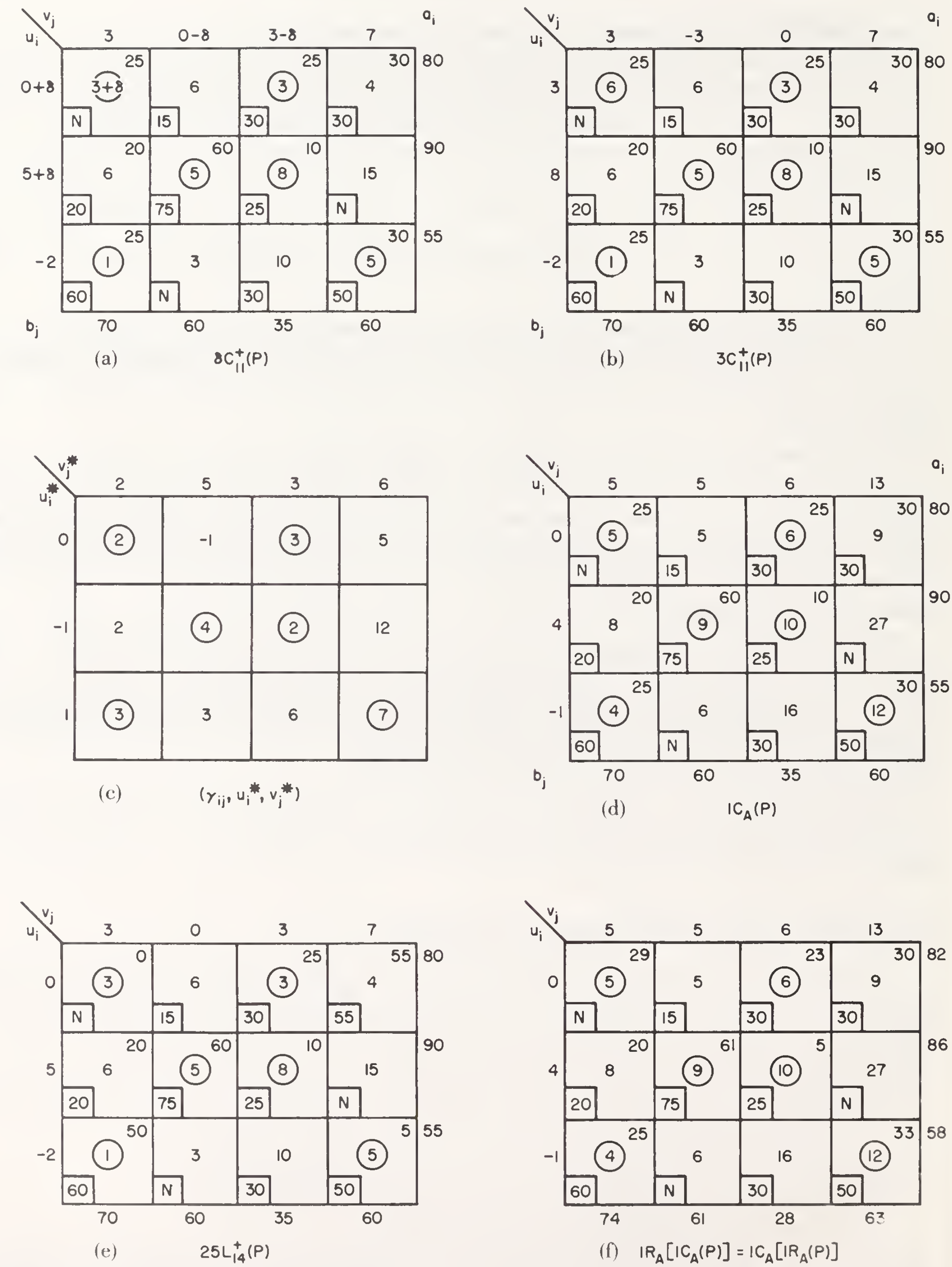

FIGURE 2.

THEOREM 9: (a) For the operator δL^+_{pq} with $(p, q) \notin UB$,

(41) $$X^+ = X,\ D^+ = D,\ Z^+ = Z \text{ and } \mu^+ = M \text{ (a large positive number)}.$$

(b) For the operator δL^-_{pq} with $(p, q) \notin UB$,

(42) $$X^- = X,\ D^- = D,\ Z^- = Z \text{ and } \mu^- = U_{pq} - x_{pq}.$$

PROOF: Since $(p, q) \notin UB$, changing U_{pq} to $U_{pq} \pm \delta$ affects neither the primal constraints (4)–(5) nor the dual feasibility conditions (12)–(14). The choice of $\mu^\pm$ in (41)–(42) is such that for $0 \leq \delta \leq \mu^\pm$ the primal feasibility condition (6) is also satisfied so that the solution is optimal. Thus $X^\pm = X$ and $D^\pm = D$. Since x_{ij} and c_{ij} have not changed, $Z^\pm = Z$.

We now consider δL^+_{pq} with $(p, q) \in UB$. In order that it be basis preserving we should set $x^+_{pq} = x_{pq} + \delta = U_{pq} + \delta = U^+_{pq}$. However, increasing x_{pq} and keeping the other x_{ij}'s same violates (4) for $i=p$ and (5) for $j=q$. To counteract this we can apply the rim operator δR^-_{pq} and modify the x_{ij}'s of the basic cells so that (4) and (5) again hold. This can be done to a maximum extent of $\mu(R^-_{pq})$ (i.e., the μ^- corresponding to the operator δR^-_{pq}). Since x_{pq} is increased the cost increases at the rate of c_{pq}, but the δR^-_{pq} operator decreases total cost at the rate of d_{pq} (Theorem 1(b)) so that the net cost decrease is at the rate of $d_{pq} - c_{pq} = w_{pq}$ (15). Similar comments apply to the δL^-_{pq} operator with $(p, q) \in UB$. However, since U_{pq} cannot be decreased below zero we have $\delta \leq U_{pq}$. We summarize the above discussion in the form of Theorem 10 below.

THEOREM 10: (a) For the operator δL^+_{pq} with $(p, q) \in UB$, $Z^+ = Z - \delta w_{pq}$; $D^+ = D$; X^+ is obtained by applying the operator δR^-_{pq} to P and then setting $x^+_{pq} = x_{pq} + \delta$; $\mu(L^+_{pq}) = \mu(R^-_{pq})$.

(b) For the operator δL^-_{pq} with $(p, q) \in UB$, $Z^- = Z + \delta w_{pq}$; and $D^- = D$; X^- is obtained by applying δR^+_{pq} to P and then setting $x^-_{pq} = x_{pq} - \delta$; $\mu(L^-_{pq}) = \text{Min}\ (U_{pq}, \mu(R^+_{pq}))$.

In a similar manner the area operator δL_A can be applied. Again to maintain the basis-preserving property, we have to set $x^A_{ij} = x_{ij} + \delta v_{ij}$ for $(i, j) \in UB$. To counteract the effect of these changes on (4) and (5) we will have to apply δR_A with $\alpha_i = -\sum_{\{j \in J | (i,j) \in UB\}} v_{ij}$ for $i \in I$ and $\beta_j = -\sum_{\{i \in I | (i,j) \in UB\}} v_{ij}$ for $j \in J$. The new optimal cost $Z^A = Z - \delta \sum_{(i,j) \in UB} v_{ij} w_{ij} = Z - \delta \sum_{i \in I} \sum_{j \in J} v_{ij} w_{ij}$ (since by (15) $w_{ij} = 0$ for $(i, j) \notin UB$). Further details are omitted.

We now consider how rim and cost operators can be simultaneously applied, i.e., we suppose that the changes in the data are as given by (17) with $v_{ij} = 0$ for $i \in I$ and $j \in J$ (which involves no loss of generality since, as shown earlier bound operators are equivalent to rim operators).

From section 4 we know that the application of rim operators affects only the primal variables. Similarly cost operators affect only dual variables so that there is no interaction effect between these two operators. In other words, $\delta T(P)$ can be applied by first applying the associated $\delta R_A(P)$ and then applying the associated $\delta C_A(P)$ on the transformed problem. We denote this operator as $\delta C_A(\delta R_A(P))$ which can be easily seen to be equal to $\delta R_A(\delta C_A(P))$. Since the final solution should be both primal and dual feasible it follows that $\mu(T(P)) = \text{Min}[\mu(R_A(P)), \mu(C_A(P))]$. The cost effect will be the sum of the two effects due to rim and cost operators, i.e., $Z^T = Z + \delta \left[\sum_{i \in I} \alpha_i u_i + \sum_{j \in J} \beta_j v_j + \sum_{i \in I} \sum_{j \in J} \gamma_{ij} x_{ij} \right]$. The above results can naturally be extended to the simultaneous application of δR^+_{pq} with δC_A or that of δR_A with δC^-_{pq}, etc.

Finally we note that a simultaneous application of a number of cell cost operators say δC^+_{ij} for $(i, j)\epsilon\Psi$ and δC^-_{ij} for $(i, j)\epsilon\chi$, where Ψ and χ are two nonintersecting subsets of $I \times J$, is equivalent to an area operator δC_A with $\gamma_{ij} = 1$ for $(i, j)\epsilon\Psi$, $\gamma_{ij} = -1$ for $(i, j)\epsilon\chi$ with $\gamma_{ij} = 0$ elsewhere. Similar comments apply to the simultaneous application of cell rim operators.

EXAMPLE (cont'd): Fig. 2(e) illustrates the result of applying the bound operator $25L^+_{14}$ to the problem P of Fig. 1(a). Since $(1, 4)\epsilon UB$ the application of this operator involves increasing x_{14} to $x_{14} + \delta$ and applying δR^-_{14} using Algorithm 2 (Theorem 10). Referring to Fig. 1(a) the cycle Γ created by adding (1, 4) to the basis is $\{(1, 4), (1, 1), (3, 1), (3, 4)\}$ with $\Gamma_1 = \{(1, 1), (3, 4)\}$ and $\Gamma_2 = \{(1, 4), (3, 1)\}$. Thus from (26) $\mu(R^-_{14}) = \text{Min}\ (x_{11}, x_{34}, U_{31} - x_{31}) = x_{11} = 25 = \mu(L^+_{14})$. Applying $25R^-_{14}$ and changing x_{14} from 30 to 55 results in the tableau of Fig. 2(e) which can be verified to be optimal. Since $w_{14} = u_1 + v_4 - c_{14} = 3$, by Theorem 10(a) $Z^+ = Z - \delta w_{14} = 945 - 75 = 870$ which agrees with $\sum_{i\epsilon I} \sum_{j\epsilon J} c_{ij}x_{ij}$ computed directly from Fig. 2(e).

Figure 2(f) illustrates the simultaneous application of the area rim operator of Fig. 1(e) and the area cost operator of Fig. 2(c). As shown earlier $\mu(R_A) = 2$ and $\mu(C_A) = 1$ so that they can be applied simultaneously to a maximum extent of $\delta = 1$ resulting in the tableau of Fig. 2(f). The reader may verify that this solution is both primal and dual feasible.

7. CONCLUSIONS

In this paper we have studied how the optimal solutions vary as the data of the problem are parametrically varied in such a way that the basis structure is preserved. In [20] we study parametric variations that can be arbitrarily large and suggest procedures for post-optimization, i.e., when a discrete change is made in the data of the problem. We also provide economic interpretations of the operators and discuss their implications for managerial decision making.

8. ACKNOWLEDGMENT

This paper and its sequel [20], resulted from a substantial revision, generalization, and unification of our earlier papers: (a) "Duality in the Transportation Model—I," Management Sciences Research Report No. 205 (presented at the 38th National Meeting of ORSA at Detroit, Oct. 1970); (b) "Duality in the Transportation Model—II", Management Sciences Research Report No. 210 (presented at the 39th National Meeting of ORSA at Dallas, May 1971); and (c) "Duality in the Transportation Model—III," Management Sciences Research Report No. 233, Graduate School of Industrial Administration, Carnegie-Mellon University, Pittsburgh, Pennsylvania. The authors wish to thank Prof. W. Szwarć for helpful comments.

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AN OPERATOR THEORY OF PARAMETRIC PROGRAMMING FOR THE TRANSPORTATION PROBLEM–II*

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ABSTRACT

This paper investigates the effect on the optimum solution of a (capacitated) transportation problem when the data of the problem (the rim conditions—i.e., the warehouse supplies and market demands—, the per unit transportation costs and the upper bounds) are continuously varied as a (linear) function of a single parameter. Operators that effect the transformation of optimum solution associated with such data changes, are shown to be a product of basis preserving operators (described in the earlier paper) that operate on a sequence of adjacent basis structures. Algorithms are provided for both rim and cost operators. The paper concludes with a discussion of the economic and managerial interpretations of the operators.

1. INTRODUCTION

In this paper we shall complete the development of an "Operator Theory" of parametric programming for the transportation problem that was initiated in [20]. The reader is assumed to be fully familiar with that reference before attempting a study of this paper.

The operators that were studied in [20] were assumed to be basis preserving (Definition 6 in [20]), i.e., the original problem P and the transformed problem P^T share a common basis structure (basis B and sets LB and UB of variables at lower and upper bounds) in their (basic) optimal solutions. In Section 2, we provide a general framework for extending the operator theory to operators that are not necessarily basis preserving. We show that an operator $\delta T(P)$ can be represented as a product basis preserving operators that operate on a sequence of "adjacent" basis structures. The detailed algorithms for applying rim and cost operators are then discussed in sections 3 and 4. In section 5 we discuss economic and managerial interpretations of the operators in determining "real" shadow prices (i.e., the rate of change of optimal cost associated with a nonzero change in the data of the problem).

In this paper we shall use the same notation, bibliography, and examples as in [20] and will continue from that paper the numbering of theorems, lemmas, remarks, equations, algorithms, figures, etc.

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2. REPRESENTATION OF AN OPERATOR AS A PRODUCT OF BASIS PRESERVING OPERATORS

Our aim in parametric programming is to determine the optimum solution of a problem P^{T} (as defined in (17)) as a function of the nonnegative parameter δ. The system of constraints for the problem P^{T} becomes

$$\sum_{j\epsilon J} x''_{ij} = a_i + \delta'' \alpha_i \qquad \text{for } i\epsilon I, \tag{43}$$

$$\sum_{i\epsilon I} x''_{ij} = b_j + \delta'' \beta_j \qquad \text{for } j\epsilon J, \text{ and} \tag{44}$$

$$0 \leqslant x''_{ij} \leqslant U_{ij} + \delta'' v_{ij} \qquad \text{for } i\epsilon I \text{ and } j\epsilon J. \tag{45}$$

The constraints (43)–(45) are linear in $\{x_{ij}\}$ and δ. Denoting by δ' the maximum value of δ (possibly ∞) for which P^{T} is feasible, we have:

LEMMA 1: The problem P^{T} is feasible over a connected region $0 \leqslant \delta \leqslant \delta' \leqslant \infty$.

Let us assume that we are interested in the optimum solution of P^{T} for δ in the range $0 \leqslant \delta \leqslant \delta^*$. Then, as shown earlier, there exists a δ' $(0 \leqslant \delta' \leqslant \delta^*)$, such that P^{T} is feasible (to be exact, primal feasible) only for the range $0 \leqslant \delta \leqslant \delta'$ (if $\delta' = \delta^*$, then the problem is feasible throughout the range of interest). The procedure we subsequently provide for applying $\delta^*\mathbf{T}(P)$ is such that it identifies δ' and finds the optimum solution for every value of δ in the range $0 \leqslant \delta \leqslant \delta'$.

Thus our method for applying $\delta^*\mathbf{T}(P)$ is basically a parametric programming procedure, but it could be used for post-optimization as well, i.e., to transform the optimum solution of P to that of a problem $\hat{P}$ that results when a discrete change is made in the data of the problem. For instance if the data of P (i.e., a_i, b_j, c_{ij}, and U_{ij}) are changed to $\hat{a}_i$, $\hat{b}_j$, $\hat{c}_{ij}$, and $\hat{U}_{ij}$ for the problem $\hat{P}$ (satisfying (A1) and (A3)), if we define

$$\alpha_i = \hat{a}_i - a_i \text{ for } i\epsilon I \text{ and } \beta_j = \hat{b}_j - b_j \text{ for } j\epsilon J,$$

$$\gamma_{ij} = \hat{c}_{ij} - c_{ij} \text{ and } v_{ij} = \hat{U}_{ij} - U_{ij} \text{ for } i\epsilon I \text{ and } j\epsilon J, \tag{46}$$

the optimum solution for $\hat{P}$ can be obtained by applying $\delta^*\mathbf{T}(P)$ with $\delta^* = 1$ (see (17)). (If there is no feasible solution to the problem $\hat{P}$, then the procedure, as mentioned earlier, would identify a number δ' with $0 \leqslant \delta' < 1$ beyond which no primal feasible solution exists.)

Consider the application of the operator $\delta^*\mathbf{T}$ to the problem $P_1 = P$ with an optimal basis structure $B_1 = B$, $LB_1 = LB$, and $UB_1 = UB$. (We shall use a subscript k to denote the kth problem P_k.) Let μ_1 be the maximum extent to which the basis preserving operator δT can be applied to P_1. (Algorithms 1 and 3 determine μ^T for a rim or cost operator. In section 6 of [20] these results were extended to other operators as well.) If $\delta^* \leqslant \mu_1$, then we are finished in the sense that the optimal solution for any $\delta (0 \leqslant \delta \leqslant \delta^*)$ can be obtained by the application of the single basis-preserving operator $\delta T(P_1)$. However if $\delta^* > \mu_1$ it is impossible to apply $\delta T(P_1)$ with $\delta > \mu_1$, because, if we did, the solution obtained will *not* be optimal (i.e., we will lose primal or dual feasibility; cf. Algorithms 2 and 4; if we applied $\delta R(P_1)$ with $\delta > \mu_1$, the leaving cell $(r, s)\epsilon B_k$ would violate (6); if we applied $\delta C(P_1)$ with $\delta > \mu_1$, the entering cell $(e, f)\epsilon LB_k$ (or UB_k) would violate dual feasibility conditions (13) (or (14))—the same logic applies to any general $\delta T(P_1)$ since in section 6 of [20] we have shown that any basis-preserving operator can be reduced to the simultaneous application of a rim and cost operator).

We denote by P_2 the problem obtained on applying $\mu_1 T$ to P_1 and by X_2, D_2, and Z_2 the associated optimal solution. From Definition 5, it directly follows that applying $\delta\mathbf{T}(P_1)$ with $\delta \geq \mu_1$ is the same as applying $(\delta-\mu_1)\mathbf{T}(P_2)$. But as mentioned earlier the operator $T(P_1)$ is applicable only up to a range of μ_1 with (B_1, LB_1, UB_1) so that $T(P_2)$ can be applied only over a zero range with (B_1, LB_1, UB_1). To apply the operator $\mathbf{T}(P_2)$ over a positive range we follow the same spirit as that of parametric programming for a general linear program, and find an *adjacent basis structure* (B_2, LB_2, UB_2) different from (B_1, LB_1, UB_1) which is also optimal for T_2 (i.e., solution X_2 (or D_2) is modified by this change, but Z_2 remains the same). In the case of rim operators a cell (r, s) was identified as the leaving cell (Algorithm 1) and in section 3, we provide the procedure for determining a nonbasic cell (e, f) so that (B_2, LB_2, UB_2) is optimal for T_2 with $B_2 = B_1 - \{(r, s)\} + \{(e, f)\}$ (it is, of course possible, that there may not exist such a cell (e, f) which will be shown to imply that $\delta' = \mu_1$ gives the maximum range over which $\delta\mathbf{T}(P)$ can be applied–i.e., the transformed problem P^T has no feasible solution for $\delta > \delta' = \mu_1$). In the case of cost operators a nonbasic cell (e, f) was identified as an entering cell (Algorithm 3) and in section 4 we provide a procedure for finding a cell (r, s) such that (B_2, LB_2, UB_2) is also optimal for T_2 with $B_2 = B_1 - \{(r, s)\} + \{(e, f)\}$. We will show in section 4 that for cost operators it is possible that $B_2 = B_1$ and the change is only in the sets LB and UB. (In the case of cost operators there is no question of primal infeasibility since changing the cost entries affects only the objective function–the optimum solution for P is, for instance, primal feasible for any other P^T.) In section 6 of [20] it was shown that any general basis preserving operator δT can be represented as $\delta T(P) = \delta R[\delta C(P)] = \delta C[\delta R(P)]$, and that $\mu(T) = \text{Min}\,[\mu(R), \mu(C)]$. The basis change for problem P_2 is to be applied based on the rim operator (cost operator) if $\mu(T) = \mu(R)$ $(\mu(T) = \mu(C))$. For the case $\mu(T) = \mu(R) = \mu(C)$ either of the basis changes can be made.

We now find μ_2, the maximum extent to which the basis preserving operator can be applied to P_2 with (B_2, LB_2, UB_2). (It is, of course, possible that $\mu_2 = 0$; but this does not imply that we cannot apply $\delta\mathbf{T}(P_2)$ with $\delta > 0$–that may be possible after obtaining some other adjacent basis structure.) If $\delta^* \leq \mu_1 + \mu_2$, then we are finished since we can apply the operator $\delta T(P_1)$ for $0 \leq \delta \leq \mu_1$, and for $\mu_1 \leq \delta \leq \mu_1 + \mu_2$ we can first obtain P_2, X_2, D_2, and Z_2 by applying $\mu_1 T(P_1)$, transform (B_1, LB_1, UB_1) to (B_2, LB_2, UB_2) (with changes in X_2 or D_2 that may be associated with this transformation) and apply $(\delta-\mu_1)T(P_2)$, i.e., apply $(\delta-\mu_1)[\mu_1 T(P_1)]$. However, if $\delta^* > \mu_1 + \mu_2$, we will have to find P_3 (by applying $\mu_2 T$ to P_2), modify (B_2, LB_2, UB_2) to (B_3, LB_3, UB_3) (if possible) with associated changes in X_3 or D_3 and find μ_3. Then for $\mu_1 + \mu_2 < \delta \leq \mu_1 + \mu_2 + \mu_3$, the relevant operator is given by

$$(\delta-\mu_1-\mu_2)T(P_3) = (\delta-\mu_1-\mu_2)[\mu_2 T(P_2)] = (\delta-\mu_1-\mu_2)[\mu_2 T(\mu_1 T(P))]$$
$$= (\delta-\mu_1-\mu_2)\prod_{k=1}^{2}\mu_k T(P).$$

Continuing in this manner we will prove that we obtain a finite sequence of "corner" problems $\{P_k\}$:

$$P = P_1, P_2, \ldots, P_k, \ldots P_{t'} \tag{47}$$

and an associated sequence of basis structures $\{(B_k, LB_k, UB_k)\}$:

$$(B, LB, UB) = (B_1, LB_1, UB_1), (B_2, LB_2, UB_2), \ldots (B_k, LB_k, UB_k), \ldots (B_{t'}, LB_{t'}, UB_{t'}), \tag{48}$$

where $(B_{t'}, LB_{t'}, UB_{t'})$ is optimal for the transformed problem P^T with $\delta=\delta'$ (as mentioned earlier P^T is feasible only for $0 \leq \delta \leq \delta'$ where $\delta' \leq \delta^*$—if $\delta'=\delta^*$ the $(B_{t'}, LB_{t'}, UB_{t'})$ is also optimal for P^T with $\delta=\delta^*$), and such that any two consecutive basis structures $(B_{k-1}, LB_{k-1}, UB_{k-1})$ and (B_k, LB_k, UB_k) are different and optimal for $P_k (2 \leq k \leq t')$. Then for any $0 \leq \delta \leq \delta'$, the operator $\delta\mathbf{T}(P)$ is represented as

(49) $$\delta\mathbf{T}(P)=\lambda T(P_t), \text{ or equivalently}$$

(50) $$\delta\mathbf{T}(P)=\lambda T[\mu_{t-1}T(\mu_{t-2}T(\ldots(\mu_1 T(P))))]=\lambda T\left[\prod_{k=1}^{t-1} \mu_k T(P)\right],$$

where $0 \leq t \leq t'$ is an integer such that

(51) $$\sum_{k=1}^{t-1} \mu_k \leq \delta \leq \sum_{k=1}^{t} \mu_k, \text{ and}$$

(52) $$\lambda=\delta-\sum_{k=1}^{t-1} \mu_k.$$

(We use the convention $\sum_{k=1}^{0} (\cdot) = 0$ and $\prod_{k=1}^{0} =$ identity operator.)

The specific details for changing the basis structurcs and the finiteness proofs will be provided in sections 3 and 4 for rim and cost operators, respectively. We make a slight change in notation and denote by X_{k+1} and D_{k+1} the optimal primal and dual solutions to T_{k+1} with $(B_{k+1}, LB_{k+1}, UB_{k+1})$. For rim operators applying $\mu_k R(P_k)$ changes X_k to X_{k+1}, but leaves D_k unaltered (see Theorem 1). In section 3 the change from (B_k, LB_k, UB_k) to $(B_{k+1}, LB_{k+1}, UB_{k+1})$ is such that the primal solution X_{k+1} remains unaltered, but the dual solution is changed from D_k to D_{k+1}. The converse is true for cost operators; applying $\mu_k C(P_k)$ changes D_k to D_{k+1}, but leaves X_k the same (Theorem 5). But the change from (B_k, LB_k, UB_k) to $(B_{k+1}, LB_{k+1}, UB_{k+1})$ in section 4 is such that the dual solution D_{k+1} remains the same, and X_k gets modified to X_{k+1}.

3. PARAMETRIC PROGRAMMING WITH RIM OPERATORS

Since most of the results of this section are common for both area and cell rim operators we use δR to denote any rim operator. Let μ_k denote the maximum extent to which the rim operator δR can be applied to the problem P_k with basis B_k. Algorithm 1, in addition to providing μ_k, determines the leaving cell $(r, s)\epsilon B_k((r_k, s_k)$ to be exact— since this depends on the iteration k—for ease in later notation, we have dropped the subscripts)—i.e., the cell that reaches either its lower or upper bound on applying $\mu_k R(P_k)$ and violates (6) if $\delta R(P_k)$ is applied with $\delta > \mu_k$. To determine the problem P_{k+1} and the optimal primal solution X_{k+1} with cost Z_{k+1}, we apply the operator $\mu_k R(P_k)$ using Algorithm 2. (The duals $u_{i,k}$ for $i\epsilon I$ and $v_{j,k}$ for $j\epsilon J$ are not affected by this transformation.) As mentioned earlier $x_{rs,k+1}=0$ or U_{rs}. We discuss below how the entering cell $\{(e, f)\}$ is determined so that the basis $B_{k+1}=B_k-\{(r, s)\}+\{(e, f)\}$ is dual feasible. This change in basis alters the dual solutions $u_{i,k}$ and $v_{j,k}$ to $u_{i,k+1}$ and $v_{j,k+1}$.

Methods for finding adjacent dual feasible bases have previously been discussed in literature in connection with the dual simplex method as applied to the transportation problem. For instance, Charnes and Cooper [2, p. 580–2] showed for a specific example how the "poly-ω" method could be used in finding an adjacent dual feasible basis. In [1] Balas and Ivanescu use a graph theoretic approach for the same. Recently Grabowski and Szwarc [12] have also provided a procedure for solving this problem. These algorithms though different in their descriptions can be shown to lead to the same adjacent dual feasible basis. We feel that the method provided in [12] is computationally more efficient than the two other approaches. The method described below is an extension of [12] for the capacitated transportation problem. Our development, however, is direct and does not require [12] for justification.

To determine an adjacent dual feasible basis we first partition the sets I and J by the cell (r, s) (see Definition 8), i.e., we define

(53) $$\Omega = B_k - \{(r, s)\},$$

and set

(54) $$\begin{aligned} I_r &= \{r\} \cup \{i\epsilon I \mid i \text{ is connected to } r \text{ in } \Omega\} \\ J_r &= \{j\epsilon J \mid j \text{ is connected to } r \text{ in } \Omega\} \\ I_s &= \{i\epsilon I \mid i \text{ is connected to } s \text{ in } \Omega\} \\ J_s &= \{s\} \cup \{j\epsilon J \mid j \text{ is connected to } s \text{ in } \Omega\}. \end{aligned}$$

REMARK 12: The "scanning routine" of Remark 9 can be used for finding the above sets (after replacing p by r and q by s in that routine). By analogy from Remark 10,

(55) $$(r, s)\epsilon[I_r \times J_s] \text{ and}$$

(56) $$B_k - \{(r, s)\} \subset [(I_r \times J_r) \cup (I_s \times J_s)].$$

If (r, s) is unique in its row (column), $J_r(I_s)$ is empty (See Remark 8).

REMARK 13: By the definition of the sets I_r, J_r, I_s, and J_s it can easily be seen that any two lines in $[I_r \cup J_r]$ are connected in $\Omega = B_k - \{(r, s)\}$. Similarly any two lines in $[I_s \cup J_s]$ are connected in Ω.

Lemma 2 below determines the set of all nonbasic cells (e, f), such that $B_k - \{(r, s)\} + \{(e, f)\}$ is a basis (Definition 2).

LEMMA 2: Let

(57) $$\Psi = [I_s \times J_r] \cup [I_r \times J_s].$$

Then $B_k - \{(r, s)\} + \{(e, f)\}$ is a basis if and only if $(e, f)\epsilon\Psi$.

PROOF: We prove this Lemma in two parts:

(a) $(e, f)\epsilon\Psi \Rightarrow B_k - \{(r, s)\} + \{(e, f)\}$ is a basis.

(b) $(e, f)\notin\Psi \Rightarrow B_k - \{(r, s)\} + \{(e, f)\}$ is not a basis.

(a) Consider any line g in $[I_s \cup J_s]$ and any line h in $[I_r \cup J_r]$. By (54) and Definition 1, g and h are not connected in $B_k - \{(r, s)\}$. Consider any cell $(e, f)\epsilon[I_s \times J_r]$. By remark 13, line g is connected

to e by a sequence S_1 (of the form (7)) in $B_k-\{(r, s)\}$. Similarly f is connected to line h by a sequence S_2 in $B_k-\{(r, s)\}$. It then directly follows that the sequence $S=S_1 \cup \{(e, f)\} \cup S_2$ connects lines g and h in $B_k-\{(r, s)\}+\{(e, f)\}$. By Remark 13 every pair of lines in $[I_r \times J_r]$ and $[I_s \times J_s]$ is connected in $B_k-\{(r, s)\}$. Consequently every pair of lines in $[I \cup J]$ is connected (uniquely) in $B_k-\{(r, s)\}+\{(e, f)\}$. Since B_k is a basis, it has $(m+n-1)$ cells. By (56), $(e, f) \notin B_k-\{(r, s)\}$. Thus, $B_k-\{(r, s)\}+\{(e, f)\}$ also has $(m+n-1)$ cells so that it is a basis. Similarly it can be shown that $(e, f) \in [I_r \times J_s] \Rightarrow B_k-\{(r, s)\}+\{(e, f)\}$ is a basis.

(b) Assume $(e, f) \notin \Psi$, then $(e, f) \in [(I_r \times J_r) \cup (I_s \times J_s)]$. By (55), $(e, f) \neq (r, s)$. Consequently (e, f) has to be nonbasic so that the basis $B_k-\{(r, s)\}+\{(e, f)\}$ will have $(m+n-1)$ cells. Consider any (e, f) in $[I_r \times J_r]$ that is nonbasic. By Remark 13, lines e and f are connected by a sequence S (say). Consequently $\{(e, f)\} \cup S$ constitutes a cycle so that $B_k-\{(r, s)\}+\{(e, f)\}$ does not constitute a basis since it will contain the same cycle. A similar proof applies when $(e, f) \in [I_s \times J_s]$.

EXAMPLE (cont'd): Considering the problem P_1 given in Fig. 1(a) of [20] with basis B_1, the application of the cell rim operator δR_{32}^{+} is shown in Fig. 1(c). It will be noted that $\mu_1=5$ since applying δR_{32}^{+} with $\delta > 5$ would violate the upper bound condition for the cell $(r, s)=(1, 3)$. The problem P_2 and the solution X_2 are shown in Fig. 1(d), where $x_{rs, 2}=x_{13, 2}=U_{13}=30$; however, the dual solutions $u_{i, 1}$ for $i \in I$ and $v_{j, 1}$ for $j \in J$ remain the same as in Fig. 1(a).

We now partition the sets I and J by $(r, s)=(1, 3)$ (using Remark 9) and obtain $I_r=\{1, 3\}$, $J_r=\{1, 4\}$, $I_s=\{2\}$, and $J_s=\{2, 3\}$. The reader may verify (55)–(56) and Remark 13 for this example. Then according to Lemma 2, the set of nonbasic cells such that $B_1-\{(1, 3)\}+\{(e, f)\}$ is a basis, is given by $\Psi=\{(2, 1), (2, 4), (1, 2), (1, 3), (3, 2), (3, 3)\}$. The reader may verify that $B_1-\{(1, 3)\}+\{(e, f)\}$, where $(e, f) \in \Psi$ does, in fact, constitute a basis (Definition 2). The only other nonbasic cell in Fig. 1(d). is $\{(1, 4)\} \in I_r \times J_r$; however, $B_1-\{(1, 3)\}+\{(1, 4)\}$ does not constitute a basis since it contains the cycle $\{(1, 1), (1, 4), (3, 4), (3, 1)\}$, thus verifying Lemma 2.

Among the adjacent bases $B_k-\{(r, s)\}+\{(e, f)\}$ with $(e, f) \in \Psi$, Theorem 11 below determines the bases (if any) that are dual feasible. Two cases arise depending on whether $x_{rs, k+1}=0$ or $=U_{rs}$.

THEOREM 11: Consider the basis B_k and the set of nonbasic cells LB_k and UB_k. Let I_r, I_s, J_r, and J_s be the result of partitioning I and J by (r, s). (See (54)). Find Δ by (i) or (ii) below depending on whether $x_{rs, k+1}=0$ or U_{rs}.

(i) $x_{rs, k+1}=0$. Set

$$\Delta = \operatorname{Min} \begin{cases} (c_{ij}-u_{i, k}-v_{j, k}) & \text{for } (i, j) \in [(I_s \times J_r) \cap LB_k] \\ (u_{i, k}+v_{j, k}-c_{ij}) & \text{for } (i, j) \in [(I_r \times J_s) \cap UB_k]. \end{cases} \tag{58}$$

(ii) $x_{rs, k+1}=U_{rs}$. Set

$$\Delta = -\operatorname{Min} \begin{cases} (c_{ij}-u_{i, k}-v_{j, k}) & \text{for } (i, j) \in [(I_r \times J_s) \cap LB_k] \\ (u_{i, k}+v_{j, k}-c_{ij}) & \text{for } (i, j) \in [(I_s \times J_r) \cap UB_k]. \end{cases} \tag{59}$$

(a) If the minimum of (58) or (59) is attained at a cell (e, f) then $B_{k+1}=B_k-\{(r, s)\}+\{(e, f)\}$ is an adjacent dual feasible basis with the sets LB_{k+1} and UB_{k+1} defined accordingly and with duals $u_{i, k+1}$ for $i \in I$ and $v_{j, k+1}$ for $j \in J$ given by (60) below:

(60)
$$u_{i,k+1} = \begin{cases} u_{i,k} - \Delta & \text{for } i\epsilon I_r \\ u_{i,k} & \text{for } i\epsilon I_s \end{cases}$$

$$v_{j,k+1} = \begin{cases} v_{j,k} + \Delta & \text{for } j\epsilon J_r \\ v_{j,k} & \text{for } j\epsilon J_s \end{cases}$$

(b) If the set over which the minimization is done is empty, then there is no primal feasible solution to the problem obtained by applying the operator $\delta\mathbf{R}(P_{k+1})$ with $\delta > 0$.

PROOF: All the proofs given below are for the case $x_{rs,k+1} = 0$. (Similar proofs can be given for the case when $x_{rs,k+1} = U_{rs}$.)

(a) Let us assume that (e, f) exists. By (55), $(r, s)\epsilon I_r \times J_s$. From (58) then $(e,f) \neq (r, s)$ since $(r, s) \notin UB_k$ so that $B_k - \{(r, s)\} + \{(e, f)\}$ is different from B_k. By (58), $(e,f)\epsilon[(I_s \times J_r) \cup (I_r \times J_s)] = \Psi$ so that by Lemma 2, $B_k - \{(r, s)\} + \{(e, f)\}$ is an adjacent basis. We now show that it is dual feasible.

For $(i, j)\epsilon B_{k+1} - \{(e, f)\}$, from (56) and (60) it follows that $u_{i,k+1} + v_{j,k+1} = u_{i,k} + v_{j,k} = c_{ij}$. The last of these equations follows since $(i,j)\epsilon B_k$ so that (12) is satisfied. For the cell $(e,f)\epsilon B_{k+1}$, two cases arise. Let us first consider the case when $(e, f)\epsilon[(I_s \times J_r) \cap LB_k]$. Then from (60) $u_{e,k+1} = u_{e,k}$ since $e\epsilon I_s$, and $v_{f,k+1} = v_{f,k} + \Delta$ since $f\epsilon J_r$. Consequently, $u_{e,k+1} + v_{f,k+1} = u_{e,k} + v_{f,k} + \Delta$. From (58), $\Delta = c_{ef} - u_{e,k} - v_{f,k}$ so that $u_{e,k} + v_{f,k} + \Delta = c_{ef}$. Hence $c_{ef} = u_{e,k+1} + v_{f,k+1}$. This last result can similarly be shown to be true when $(e,f)\epsilon[(I_r \times J_s) \cap UB_k]$. Thus the dual solutions $u_{i,k+1}$ for $i\epsilon I$ and $v_{j,k+1}$ for $j\epsilon J$ satisfy (12) for all $(i,j)\epsilon B_{k+1}$.

For the nonbasic cells $(i, j)\epsilon[I_r \times J_r] \cup [I_s \times J_s]$, $u_{i,k+1} + v_{j,k+1} = u_{i,k} + v_{j,k}$ from (60), so that the dual feasibility conditions (13) and (14) are satisfied with respect to these cells for B_{k+1} since they are satisfied for B_k.

All the remaining nonbasic cells are in $[I_s \times J_r] \cup [I_r \times J_s]$. For

$$(i, j)\epsilon I_s \times J_r,\ u_{i,k+1} + v_{j,k+1} = u_{i,k} + v_{j,k} + \Delta.$$

Since B_k is dual feasible, $\Delta \geqslant 0$ in (58). For

$$(i, j)\epsilon[(I_s \times J_r) \cap UB_k],\ u_{i,k+1} + v_{j,k+1} = u_{i,k} + v_{j,k} + \Delta \geqslant c_{ij}$$

since $u_{i,k} + v_{j,k} \geqslant c_{ij}$ for B_k and $\Delta \geqslant 0$ thus satisfying (14) for these cells. For $(i,j)\epsilon[(I_s \times J_r) \cap LB_k]$, $u_{i,k+1} + v_{j,k+1} = u_{i,k} + v_{j,k} + \Delta \leqslant c_{ij}$ by the choice of Δ in (58), thus satisfying (13) for these cells. Similarly (13) and (14) are satisfied by the nonbasic cells in $[I_r \times J_s]$. Thus the basis

$$B_{k+1} = B_k - \{(r,s)\} + \{(e, f)\}$$

is dual feasible.

(b) We now consider the case when the sets over which the minimization is done is empty. We show below that this implies that there is no primal feasible solution to any problem obtained by applying the operator $\delta\mathbf{R}(P_{k+1})$ with $\delta > 0$ (i.e., $(\mu_k + \delta)\mathbf{R}(P_k)$). We prove this by constructing a feasible solution to the corresponding dual problem (9)–(11) and showing that its objective function (9) is unbounded. Then by the familiar duality results in linear programming [6, 15] the assertion that there is no primal feasible solution holds true.

Consider the application of the operator $\delta R(P_{k+1})$ where δ is strictly positive (which is same as applying $(\mu_k + \delta)R(P_k)$). Let us denote by $\hat{a}_i$ and $\hat{b}_j$ the rims for the problem $\hat{P}$ obtained when $\delta = \hat{\delta} > 0$.

Applying $(\mu_k+\hat{\delta})R(P_k)$ using Algorithm 2 would result in $\hat{x}_{rs}<0$ (some other $\hat{x}_{ij}$ for $(i,j)\epsilon B_k$, may also violate (6) depending on the magnitude of $\hat{\delta}$). We note that the application of this operator does not affect the amounts x_{ij} corresponding to nonbasic cells (see Algorithm 2). Since the set over which the minimization (58) is done is assumed to be empty (and since (r, s) is the only basic cell in $[(I_r\times J_s)\cup(I_s\times J_r)]$ – see (55)–(56)) we can derive the following conditions:

(61) $$\hat{x}_{ij}=U_{ij} \qquad \text{for } (i,j)\epsilon[I_s\times J_r] \text{ and}$$

(62) $$\hat{x}_{ij}=0 \qquad \text{for } (i,j)\epsilon[(I_r\times J_s)-\{(r,s)\}].$$

We now define

(63) $$\sum_{i\epsilon I_r}\sum_{j\epsilon J_r}\hat{x}_{ij}=K \text{ (say)}.$$

From (62), (63), and (4), we obtain

(64) $$\sum_{i\epsilon I_r}\sum_{j\epsilon J}\hat{x}_{ij}=\sum_{i\epsilon I_r}\sum_{j\epsilon J_r}\hat{x}_{ij}+\sum_{i\epsilon I_r}\sum_{j\epsilon J_s}\hat{x}_{ij}=K+\hat{x}_{rs}=\sum_{i\epsilon I_r}\hat{a}_i.$$

Similarly from (61), (63), and (5), we get

(65) $$\sum_{i\epsilon I}\sum_{j\epsilon J_r}\hat{x}_{ij}=\sum_{i\epsilon I_r}\sum_{j\epsilon J_r}\hat{x}_{ij}+\sum_{i\epsilon I_s}\sum_{j\epsilon J_r}\hat{x}_{ij}=K+\sum_{i\epsilon I_s}\sum_{j\epsilon J_r}U_{ij}=\sum_{j\epsilon J_r}\hat{b}_j.$$

From (64) and (65), we get

(66) $$\sum_{i\epsilon I_r}\hat{a}_i-\sum_{j\epsilon J_r}\hat{b}_j+\sum_{i\epsilon I_s}\sum_{j\epsilon J_r}U_{ij}=\hat{x}_{rs}<0.$$

We now assert that the solutions $u_{i,k+1}$ and $v_{j,k+1}$ given by (60) are dual feasible ((10) and (11)) for the problem $\hat{P}$ regardless of how large Δ may be. To prove this assertion we consider the conditions (10) and (11) for the mutually exclusive and collectively exhaustive subsets (i) $[(I_r\times J_r)\cup(I_s\times J_s)]$, (ii) $[I_r\times J_s]$, and (iii) $[I_s\times J_r]$.

(i) $(i,j)\epsilon[(I_r\times J_r)\cup(I_s\times J_s)]$: From (60), $u_{i,k+1}+v_{j,k+1}=u_{i,k}+v_{j,k}$ so that by setting $w_{ij,k+1}=w_{ij,k}$ the dual feasibility conditions are satisfied for these cells since they are for basis B_k.

(ii) $(i,j)\epsilon[I_r\times J_s]$: From (62) and since B_k is dual feasible, $u_{i,k}+v_{j,k}\leqslant c_{ij}$ (this holds for the cell $(r,s)\epsilon B_k$ by (12)). From (15) $w_{ij,k}=0$. By (60), $u_{i,k+1}+v_{j,k+1}=u_{i,k}+v_{j,k}-\Delta$. Since Δ is assumed to be $\geqslant 0$ by setting $w_{ij,k+1}=0$, conditions (10) and (11) are satisfied by the new dual solutions.

(iii) $(i,j)\epsilon[I_s\times J_r]$: From (61) and since B_k is dual feasible, $u_{i,k}+v_{j,k}\geqslant c_{ij}$. From (15), $w_{ij,k}=u_{i,k}+v_{j,k}-c_{ij}$. From (60), $u_{i,k+1}+v_{j,k+1}=u_{i,k}+v_{j,k}+\Delta$. By setting $w_{ij,k+1}=w_{ij,k}+\Delta$, (10) and (11) are again satisfied.

Thus the solution $(u_{i,k+1}, v_{j,k+1}, w_{ij,k+1})$ satisfies (10) and (11). The value of the objective function (9) for this solution is:

$$\hat{F}=\sum_{i\epsilon I}\hat{a}_iu_{i,k+1}+\sum_{j\epsilon J}\hat{b}_jv_{j,k+1}-\sum_{i\epsilon I}\sum_{j\epsilon J}U_{ij}w_{ij,k+1}$$

$$= \left[\sum_{i \epsilon I} \hat{a}_i u_{i,k} + \sum_{j \epsilon J} \hat{b}_j v_{j,k} - \sum_{i \epsilon I} \sum_{j \epsilon J} U_{ij} w_{ij,k} \right]$$

$$- \Delta \sum_{i \epsilon I_r} \hat{a}_i + \Delta \sum_{j \epsilon J_r} \hat{b}_j - \Delta \sum_{i \epsilon I_s} \sum_{j \epsilon J_r} U_{ij},$$

calling the expression within the square brackets F_0, we get

$$\hat{F} = F_0 - \Delta \left[\sum_{i \epsilon I_r} \hat{a}_i - \sum_{j \epsilon J_r} \hat{b}_j + \sum_{i \epsilon I_s} \sum_{j \epsilon J_r} U_{ij} \right]. \tag{67}$$

From (66) and (67)

$$\hat{F} = F_o - \Delta \hat{x}_{rs}. \tag{68}$$

Since $\hat{x}_{rs} < 0$ the dual objective function $\hat{F}$ can be made as large as desired by increasing Δ, i.e., the objective function (9) is unbounded.

From linear programming duality theory [6, 15] it then follows that the problem defined by $\delta\mathbf{T}(P_{k+1})$ with $\delta > 0$ has no primal feasible solution, thus completing the proof.

The next two lemmas determine the effect of the basis change from B_k to B_{k+1} on the cost effects of the cell and area rim operators—i.e., on the expressions $\{d_{pq}\}$ and $\left(\sum_{i \epsilon I} \alpha_i u_i + \sum_{j \epsilon J} \beta_j v_j \right)$ (see Theorem 1).

LEMMA 3:
(a) For the operator $\delta\mathbf{R}^+_{pq}$, $(p, q) \epsilon [I_s \times J_r]$ if $x_{rs,k+1} = 0$ and $(p, q) \epsilon [I_r \times J_s]$ if $x_{rs,k+1} = U_{rs}$.
(b) For the operator $\delta\mathbf{R}^-_{pq}$, $(p, q) \epsilon [I_r \times J_s]$ if $x_{rs,k+1} = 0$ and $(p, q) \epsilon [I_s \times J_r]$ if $x_{rs,k+1} = U_{rs}$.
(c) For the operator $\delta\mathbf{R}_A$, $\left(\sum_{i \epsilon I_r} \alpha_i - \sum_{j \epsilon J_r} \beta_j \right) < 0$ if $x_{rs,k+1} = 0$, and $\left(\sum_{i \epsilon I_r} \alpha_i - \sum_{j \epsilon J_r} \beta_j \right) > 0$ if $x_{rs,k+1} = U_{rs}$.

PROOF: (a) We first consider the case when $x_{rs,k+1} = 0$. From (19) and (20), it follows that $(p, q) \notin B$. From (24), $(r, s) \epsilon [\Gamma_2 - \{(p, q)\}]$, i.e., when (p, q) is added to B_k, a cycle Γ is created with an *odd* number of basis cells in the cycle between (p, q) and (r, s) (see Definition 3).

We now show that $(p, q) \notin [(I_r \times J_r) \cup (I_s \times J_s)]$. To prove this assume the contrary, say $(p, q) \epsilon [I_r \times J_r]$. From Remark 13 there exists a sequence S (of the form (7)) connecting row p to column q in $B_k - \{(r, s)\}$. By Definition 2 $\{(p, q)\} \cup S$ constitutes the cycle. Since $S \subset [B_k - \{(r, s)\}]$ (r, s) does *not* belong to the cycle, a contradiction. A similar proof would show that $(p, q) \notin (I_s \times J_s)$ either.

To show that $(p, q) \notin [I_r \times J_s]$ again assume the contrary, i.e., $p \epsilon I_r$. By definition of the set I_r (54) and from Remark 2, it follows that there are an *even* number of cells in the path between (p, q) and (r, s) which is a contradiction. Thus $(p, q) \epsilon [I_s \times J_r]$.

A similar proof holds for the case when $x_{rs,k+1} = U_{rs}$. (In this case there is the additional possibility of $(p, q) \epsilon B$ (Theorem 2) so that $(p, q) = (r, s)$. By (55) then $(p, q) \epsilon [I_r \times J_s]$).

The proof for (b) is similar.

(c) By Definition 7, $y_{ij} = 0$ for nonbasic cells so that from (56) we obtain,

$$y_{ij} = 0 \qquad \text{for } (i, j) \epsilon [(I_r \times J_s) - \{(r, s)\}] \cup [I_s \times J_r]. \tag{69}$$

We now define

$$\sum_{i\epsilon I_r} \sum_{j\epsilon J_r} y_{ij} = K. \tag{70}$$

From (27), (69), and (70)

$$\sum_{i\epsilon I_r} \sum_{j\epsilon J} y_{ij} = \sum_{i\epsilon I_r} \sum_{j\epsilon J_r} y_{ij} + \sum_{i\epsilon I_r} \sum_{j\epsilon J_s} y_{ij} = K + y_{rs} = \sum_{i\epsilon I_r} \alpha_i. \tag{71}$$

Similarly from (28), (69), and (70)

$$\sum_{i\epsilon I} \sum_{j\epsilon J_r} y_{ij} = \sum_{i\epsilon I_r} \sum_{j\epsilon J_r} y_{ij} + \sum_{i\epsilon I_s} \sum_{j\epsilon J_r} y_{ij} = K = \sum_{j\epsilon J_r} \beta_j. \tag{72}$$

From (71) and (72)

$$\sum_{i\epsilon I_r} \alpha_i - \sum_{j\epsilon J_r} \beta_j = y_{rs}. \tag{73}$$

From (29)–(30) and Algorithms 1 and 2, $y_{rs} < 0$ for the case $x_{rs,k+1} = 0$ and $y_{rs} > 0$ for the case $x_{rs,k+1} = U_{rs}$ thus proving (c).

LEMMA 4:

(a) For the operator $\delta\mathbf{R}^+_{pq}$, $d_{pq,k+1} \geqslant d_{pq,k}$.

(b) For the operator $\delta\mathbf{R}^-_{pq}$, $d_{pq,k+1} \leqslant d_{pq,k}$.

(c) For the operator $\delta\mathbf{R}_A$, $\sum_{i\epsilon I} \alpha_i u_{i,k+1} + \sum_{j\epsilon J} \beta_j v_{j,k+1} \geqslant \sum_{i\epsilon I} \alpha_i u_{i,k} + \sum_{j\epsilon J} \beta_j v_{j,k}$.

PROOF:

(a) Consider the case $x_{rs,k+1} = 0$. By Lemma 3(a), $(p,q)\epsilon[I_s \times J_r]$ so that from (60),

$$d_{pq,k+1} = u_{p,k+1} + v_{q,k+1} = u_{p,k} + v_{q,k} + \Delta = d_{pq,k} + \Delta.$$

From (58) and from the fact that B_k is dual feasible, $\Delta \geqslant 0$. Thus $d_{pq,k+1} \geqslant d_{pq,k}$. The same result can be similarly shown to be true when $x_{rs,k+1} = U_{rs}$.

(b) This proof is similar to that of (a).

(c) Let us consider the case when $x_{rs,k+1} = 0$. From (60)

$$\left(\sum_{i\epsilon I} \alpha_i u_{i,k+1} + \sum_{j\epsilon J} \beta_j v_{j,k+1}\right) - \left(\sum_{i\epsilon I} \alpha_i u_{i,k} + \sum_{j\epsilon J} \beta_j v_{j,k}\right) = -\Delta\left(\sum_{i\epsilon I_r} \alpha_i - \sum_{j\epsilon J_r} \beta_j\right). \tag{74}$$

From (58) and from the fact that B_k is dual feasible, $\Delta \geqslant 0$. By Lemma 3(c), $\sum_{i\epsilon I_r} \alpha_i - \sum_{j\epsilon J_r} \beta_j < 0$. Thus the expression on the left hand side of Equation (74) is $\geqslant 0$. A similar proof holds when $x_{rs,k+1} = U_{rs}$.

We summarize the results of this section by the following algorithm.

ALGORITHM 5: For applying the operator $\delta^*\mathbf{R}(P)$.

(i) Let $k=1, P_1=P, B_1=B, LB_1=LB, UB_1=UB, X_1=X, Z_1=Z, u_{i,1}=u_i$ for $i\epsilon I$, and $v_{j,1}=v_j$ for $j\epsilon J$.

(ii) Using Algorithm 1 determine μ_k and the leaving cell (r, s). If $\sum_{i=1}^{k}\mu_i \geq \delta^*$, set $\delta'=\delta^*$, $t'=k$ and go to (iv). Otherwise apply the operator $\mu_k R(P_k)$ using Algorithm 2. Define the resulting problem P_{k+1} with optimal primal solution X_{k+1} and cost Z_{k+1} (the dual variables $u_{i,k}$ and $v_{j,k}$ are not altered by this transformation).

(iii) Partition the sets I and J by the cell (r, s) (see Equation (54)) using Remark 9 into I_r, I_s, J_r, and J_s. Use (58) or (59) (depending on whether $x_{rs,k+1}=0$ or U_{rs}) to determine Δ and the cell (e, f) (if any). If no such cell (e, f) exists (i.e., the set over which the minimization is done is empty) set $\delta'=\sum_{i=1}^{k}\mu_i$, $t'=k$ and go to (iv). Otherwise set $B_{k+1}=B_k-\{(r, s)\}+\{(e, f)\}$. Obtain the sets LB_{k+1} and UB_{k+1} accordingly. Determine the duals $u_{i,k+1}$ for $i\epsilon I$ and $v_{j,k+1}$ for $j\epsilon J$ using (60). Set $k=k+1$ and go to (ii).

(iv) The problem P^T defined by the operator $\delta\mathbf{R}(P)$ is primal feasible over the range $0 \leq \delta \leq \delta'$. To find the optimum solution for any δ in this range, determine t and λ as defined by (51) and (52) and apply $\lambda R(P_t)$ using Algorithm 2. STOP.

REMARK 14: We now derive an expression for the optimal cost Z as a function of δ in the range $0 \leq \delta \leq \delta'$. By (49)–(52) and Algorithm 2, it follows that

$$Z^+(\delta)=Z_1+\sum_{k=1}^{t-1}\mu_k d_{pq,k}+\lambda d_{pq,t} \qquad \text{for the operator } \delta\mathbf{R}^+_{pq}, \tag{75}$$

$$Z^-(\delta)=Z_1-\sum_{k=1}^{t-1}\mu_k d_{pq,k}-\lambda d_{pq,t} \qquad \text{for the operator } \delta\mathbf{R}^-_{pq}, \text{ and} \tag{76}$$

$$Z^A(\delta)=Z_1+\left[\sum_{k=1}^{t-1}\mu_k\left(\sum_{i\epsilon I}\alpha_i u_{i,k}+\sum_{j\epsilon J}\beta_j v_{j,k}\right)\right]+\lambda\left(\sum_{i\epsilon I}\alpha_i u_{i,t}+\sum_{j\epsilon J}\beta_j v_{j,t}\right) \tag{77}$$

for the operator $\delta\mathbf{R}_A$, where t and λ are as defined by (51)–(52).

Thus the cost curves are piecewise linear. From Lemma 4 and Equations (75)–(77), the cost function $Z(\delta)$ can be seen to be convex, a well known result in parametric programming (of the "Requirements vector") for general linear programming problems.

EXAMPLE (cont'd): We continue in Fig. 3 the application of the same rim operators discussed earlier (see Figure 1(c)–(f)). Let us assume that we are interested in the optimum solution of the transformed problem for all values of δ, $0 \leq \delta \leq \delta^*=1000$ (say).

Consider the application of the operator $\delta^*\mathbf{R}^+_{32}$ (P) (using Algorithm 5), where $P_1=P$ is the problem of Fig. 1(a) with optimum solution X_1, Basis B_1, duals $u_{i,1}$ and $v_{j,1}$, and cost $Z_1=945$. We showed in section 3 of [20] that application of Algorithm 1 for the operator δR^+_{32} can be done to a maximum extent of $\mu_1=5$ with $(r, s)=(1, 3)$ as the leaving cell. (The optimum solution for $0 \leq \delta \leq 5$ is shown in Fig. 1(c) with a cost effect of $d_{32,1}=-2$.) In Step (ii) of Algorithm 5, we find $\mu_1=5<\delta^*=1,000$ so that we apply $5R^+_{32}$ (P_1) (using Algorithm 2) to obtain the problem P_2 shown in Fig. 1(d) with optimal primal solution X_2 and cost $Z_2=Z_1+\mu_1 d_{32}=935$.

In Step (iii) we partition I and J by $(r, s)=(1, 3)$ to get $I_r=\{1,3\}$, $J_r=\{1, 4\}$, $I_s=\{2\}$ and $J_s=\{2, 3\}$. Since $x_{rs,2}=U_{rs}=30$, we use the minimization (59) to obtain (e, f) and Δ. For this problem $[I_r\times J_s]=\{(1, 2), (1, 3), (3, 2), (3, 3)\}$. Of these all the cells are nonbasic with $x_{ij}=0$ except that $(1, 3)\epsilon B_1$. Consequently $[(I_r\times J_s)\cap LB_1]$ is $\{(1, 2), (3, 2), (3, 3)\}$. Similarly $[I_s\times J_r]=\{(2, 1), (2, 4)\}$. Since $x_{21}=U_{21}=20$ and $x_{24}=0\neq U_{24}$, $[I_s\times J_r)\cap UB_1]=\{(2, 1)\}$. Thus minimization (59) yields:

$$\Delta=-\text{Min}\,[\{(6-0-0), (3+2-0), (10+2-3)\}, \{(5+3-6)\}]=-2$$

with $(e, f)=(2, 1)$. Consequently, we obtain the adjacent basis $B_2=B_1-\{(1, 3)\}+\{(2, 1)\}$, as shown in Fig. 3(a) with $UB_2=UB_1+\{(1, 3)\}-\{(2, 1)\}$. The duals $u_{i,2}$ and $v_{j,2}$ obtained by using (60) are shown on the left and top rims of this tableau. The reader may verify that the primal solution and cost remain unaffected by this basis change and that the dual solution of Fig. 3(a) is dual feasible (i.e., satisfies (12)–(14) with basis structure (B_2, LB_2, UB_2). Lemma 3(a) holds in this case since $(p, q)=(3, 2)$ $\epsilon\,[I_r\times J_s)$. Also $d_{32,2}=u_{3,2}+v_{2,2}=0>d_{32,1}=-2$ thus verifying Lemma 4(a).

We now go back to Step (ii) of Algorithm 5 with $k=2$. In applying Algorithm 1 to find μ_2 corresponding to $\delta R^+_{32}(P_2)$, we get the cycle Γ shown in Fig. 3(b) so that $\mu_2=10$ with $(r, s)=(2, 2)$ as the leaving cell. The optimum solution for $5\leqslant\delta\leqslant 15=\mu_1+\mu_2$ is shown in Fig. 3(b) where $\lambda=\delta-\mu_1=\delta-5$ (see (51) and (52)). Since $\mu_1+\mu_2=15<\delta^*=1{,}000$, we apply $10R^+_{32}(P_2)$ to obtain P_3 shown in Fig. 3(c) with solution X_3 and cost $Z_3=Z_2+\mu_2 d_{32,2}=935$. We now partition I and J by $(r, s)=(2, 2)$ to obtain $I_r=\{1, 2, 3\}$, $J_r=\{1, 3, 4\}$, $I_s=\phi$ and $J_s=\{2\}$ (see Remark 8). Since $x_{rs,3}=x_{22,3}=75=U_{22}$, we apply (59). Now $[(I_r\times J_s)\cap LB_2]=\{(1, 2), (3, 2)\}$ and $I_s\times J_r=\phi$ so that $\Delta=-\text{Min}\,\{(6-2-0), (3-0-0)\}=-3$ and $(e, f)=(3, 2)$. The basis $B_3=B_2-\{(2, 2)\}+\{(3, 2)\}$ is shown in Fig. 3(d) with the duals calculated using (60). Again the reader may verify that the solution is dual feasible and that Lemma 3(a) holds. Since $d_{32,3}=3>d_{32,2}=0$, Lemma 4(a) is easily verified.

When we go back to Step (ii) of Algorithm 5, we find $\mu_3=N$ since $(3, 2)\epsilon B_3$. Thus $\sum_1^3\mu_i=N+15>\delta^*=1{,}000$ and we go to (iv). The optimal solution for $\delta\geqslant 15$ is shown in Fig. 3(d) where $\lambda=\delta-\mu_1-\mu_2=\delta-15$. For this problem the optimal cost Z^+ as a function of δ can be written as $Z^+=\{(945-2\delta)$ for $0\leqslant\delta\leqslant 5$; 935 for $5\leqslant\delta\leqslant 15$; $935+3\ (\delta-15)$ for $\delta\geqslant 15\}$ which is clearly piecewise linear and convex (slope increases from -2 to 0 to 3) thus verifying Remark 14 for this case.

We now consider the area rim operator $\delta^*\mathbf{R}_A(P)$ shown in Fig. 1(e) with $\delta^*=1{,}000$ (say). In section 4 of [20] we showed that the basis preserving operator $\delta R_A(P_1)$ can be applied to a maximum extent of $\mu_1=2$ with $(r, s)=(2, 3)$ as the leaving cell (Fig. 1(e) and (f)). The cost effect of this operator $\sum_{i\epsilon I}\alpha_i u_{i,1}+\sum_{j\epsilon J}\beta_j v_{j,1}=-14$. Since $\mu_1<\delta^*$ we apply $2R_A(P_1)$ to obtain the solution X_2 shown in Fig. 1(f) with cost $Z_2=945-(2\times 14)=917$.

In Step (iii) of Algorithm 5 we partition I and J by $(r, s)=(2, 3)$ to get $I_r=\{2\}$, $J_r=\{2\}$, $I_s=\{1, 3\}$, $J_s=\{1, 3, 4\}$. Since $x_{rs,2}=0$ we use (58) to obtain $(e, f)=(2, 1)$ and $\Delta=2$. The basis $B_2=B_1-\{(2, 3)\}+\{(2, 1)\}$ is shown in Fig. 3(e) and (f). The duals modified as per (60) are shown on the rims of Fig. 3(f) and can be verified to satisfy (12)–(14). Lemma 3(c) holds since $\sum_{i\epsilon I_r}\alpha_i-\sum_{j\epsilon J_r}\beta_j=-5$. The cost effect $\sum_{i\epsilon I}\alpha_i u_{i,2}+\sum_{j\epsilon J}\beta_j v_{j,2}=-4>-14$ at the previous iteration thus verifying Lemma 4(c). We now go back to Step (ii) of Algorithm 5 and determine μ_2. The values y_{ij} needed in Algorithm 1 are shown

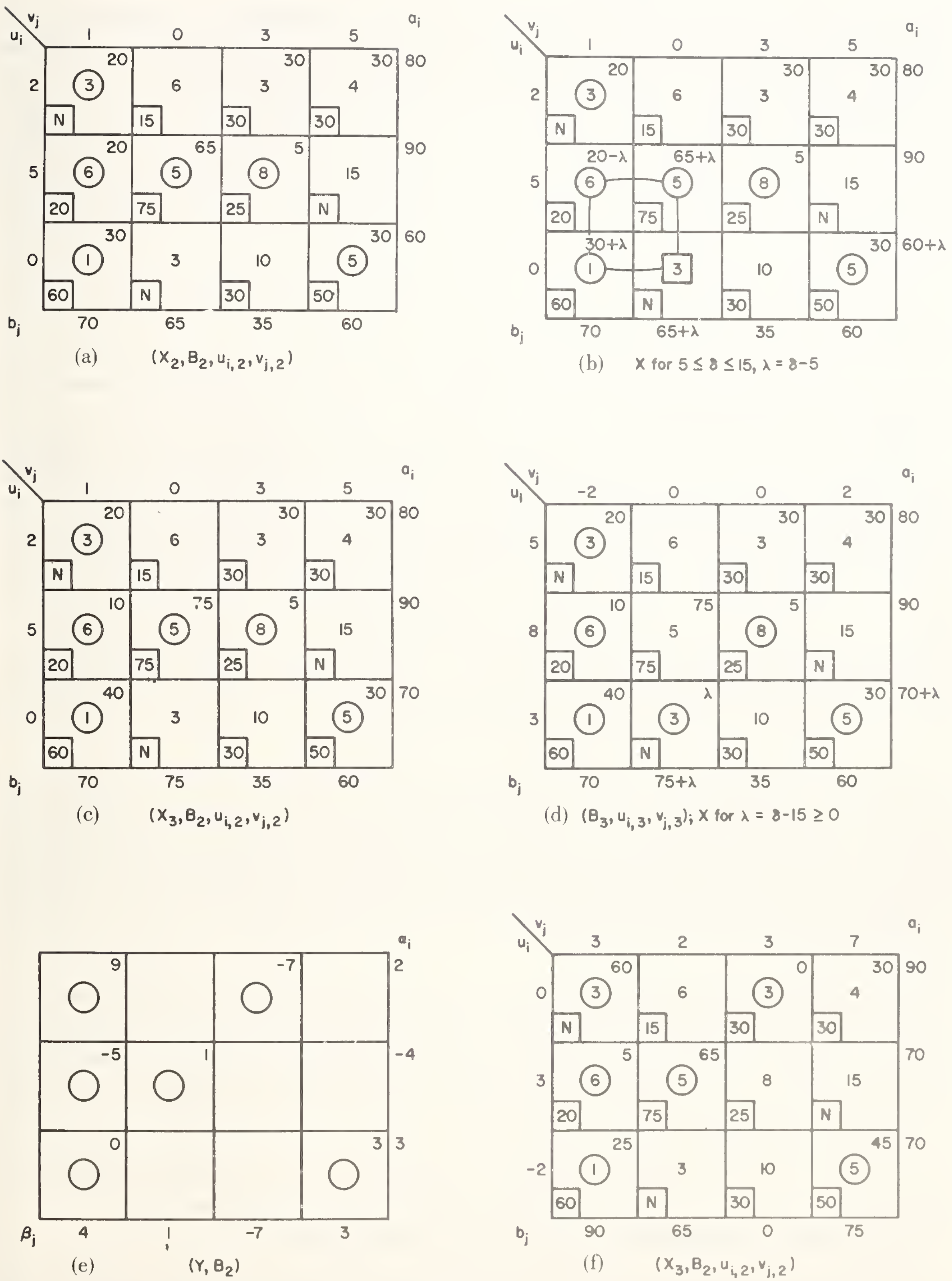

(a) $(X_2, B_2, u_{i,2}, v_{j,2})$
(b) X for $5 \le \delta \le 15$, $\lambda = \delta - 5$
(c) $(X_3, B_2, u_{i,2}, v_{j,2})$
(d) $(B_3, u_{i,3}, v_{j,3})$; X for $\lambda = \delta - 15 \ge 0$
(e) (Y, B_2)
(f) $(X_3, B_2, u_{i,2}, v_{j,2})$

FIGURE 3

in Fig. 3(e), so that $\mu_2=3$ with $(r, s)=(1, 3)$. Since $\mu_1+\mu_2=5<1{,}000=\delta^*$, we apply $3R_A(P_2)$ to obtain P_3 shown in Fig. 3(f) with solution X_3 and cost $Z_3=Z_2-(4\times 3)=905$. The optimum solution for $2\leqslant\delta\leqslant 5$ can be obtained by applying $\lambda R_A(P_2)$ with $\lambda=\delta-2$ (using Algorithm 2).

Partitioning I and J by $(r, s)=(1, 3)$, we obtain $I_r=\{1, 2, 3\}$, $J_r=\{1, 2, 4\}$, $I_s=\phi$, $J_s=\{3\}$. Since $x_{rs,2}=x_{13,2}=0$ we minimize using (58). But $I_s\times J_r=\phi$ and $[(I_r\times J_s)\cap UB_2]=\phi$ so that there is no cell (e,f) with which we can obtain an adjacent basis structure. In Fig. 3(f) $b_{3,2}=0$. Since $\beta_3=-7$ this explains why we are unable to find any primal feasible solution for $\lambda=\delta-5>0$ ($b_3^T=b_3-\delta\beta_3=35-7\delta$ becomes negative for $\delta>5$ violating (A1) and (A2)). The reader may verify the proof of Theorem 11 by taking $\delta=6$.

Thus the area operator $\delta\mathbf{R}_A$ leads to a transformed problem that is primal feasible for $0\leqslant\delta\leqslant 5$ and infeasible for $\delta>5$. For this problem the cost Z as a function of δ is $\{945-14\delta$ for $0\leqslant\delta\leqslant 2$, $917-4(\delta-2)$ for $2\leqslant\delta\leqslant 5\}$. $Z^A(\delta)$ is piecewise linear and convex thus verifying Remark 14.

To prove that Algorithm 5 terminates in a finite number of steps, we need an assumption that the problem P is *dual nondegenerate.*

DEFINITION 10: The problem P defined in (3)–(6) is *dual nondegenerate*, if corresponding to every dual feasible basis B with associated duals u_i for $i\epsilon I$ and v_j for $j\epsilon J$ satisfying (12), conditions (13) and (14) are satisfied as strict inequalities.

REMARK 15: By perturbing the cost coefficients c_{ij} (which are assumed to be scaled, if necessary, to be integers) slightly as

$$c'_{ij}=c_{ij}+\zeta\theta^{[n(i-1)+j]} \text{ for } i\epsilon I \text{ and } j\epsilon J \tag{78}$$

where θ is an integer >2 and ζ is chosen so that

$$0<\zeta<(\theta-1)/[2(\theta^{mn+1}-1)] \tag{79}$$

the problem P can be proved to be dual nondegenerate [12]. The upper bound in (79) is such that rounding $(c'_{ij}-u'_i-v'_j)$ to the nearest integer will give its "true" value, i.e., when c'_{ij} is replaced by c_{ij}. From a computational point of view setting θ as an integer >2 ($\theta=3$ say) is not very practical for large m and n (>50, say). Choosing θ to be an arbitrary number just a little larger than 1 will make the problem almost always dual nondegenerate.

THEOREM 12: Assume that P is dual nondegenerate. Then Algorithm 5 terminates in a finite number of steps.

PROOF: From the nature of Algorithm 5, to prove that this theorem is true it is enough if we can show that no basis will be repeated (since the total number of bases is finite). By the nondegeneracy assumption, the value of Δ found in (58) or (59) is nonzero. From the proofs of Lemma 4, it then follows that the cost factors d_{pq}, $-d_{pq}$ and $\sum_{i\epsilon I}\alpha_i u_i+\sum_{j\epsilon J}\beta_j v_j$ are monotone increasing functions (of k, the iteration number) for the operators δR^+_{pq}, δR^-_{pq}, δR_A, respectively. Since d_{pq} or $\left(\sum_{i\epsilon I}\alpha_i u_i+\sum_{j\epsilon J}\beta_j v_j\right)$ is for a basis (see (12) and Remark 6), no basis will be repeated thus completing the proof.

4. PARAMETRIC PROGRAMMING WITH COST OPERATORS

Since the initial problem P was assumed to be primal feasible (see Assumption A2), and since cost operators affect only the cost entries, it is obvious that after applying any cost operator the trans-

formed problem is primal feasible for the entire range $0 \leq \delta \leq \delta^*$. Since most of the results of this section are common for both area and cell cost operators, we use δC to denote any cost operator. Let μ_k be the maximum extent to which a basis preserving operator δC can be applied to the problem P_k with basis structure (B_k, LB_k, UB_k). Algorithm 3 determines μ_k and the entering cell (e, f) (the subscripts k on (e_k, f_k) have been dropped to avoid confusing notation later). The amount μ_k is such that the nonbasic cell (e, f) would violate the dual feasibility condition (13) or (14) if the basis preserving operator $\delta C(P_k)$ is applied with $\delta > \mu_k$. Let P_{k+1} be the problem (with optimum dual solution D_{k+1} and cost Z_{k+1}) that results on applying $\mu_k C$ to P_k using Algorithm 4 (the primal solution X_k remains unaltered during this transformation). By the definition of μ_k and the choice of cell (e, f) it follows that

(80) $$u_{e,\,k+1} + v_{f,\,k+1} = c_{ef,\,k+1}\,.$$

Thus the cell (e, f) satisfies condition (12)–(14) (though it is nonbasic) and hence qualifies as a basic cell or for a change from LB_k to UB_{k+1} or from UB_k to LB_{k+1}. We describe below how a cell (r, s) belonging to the current basis B_k is chosen in such a way that $B_{k+1} = B_k - \{(r, s)\} + \{(e, f)\}$ is also an optimal basis for the problem P_{k+1}. Alternatively, the only change in going from (B_k, LB_k, UB_k) to $(B_{k+1}, LB_{k+1}, UB_{k+1})$ may be that (e, f) changes from LB_k to UB_{k+1} or from UB_k to LB_{k+1}. Since (e, f) already satisfies (12), the duals $u_{i,\,k+1}$ for $i \epsilon I$ and $v_{j,\,k+1}$ for $j \epsilon J$ remain the same and $(B_{k+1}, LB_{k+1}, UB_{k+1})$ is dual feasible. Consequently, the change in basis structure is to be carried out in such a way that the primal solution X_{k+1} associated with $(B_{k+1}, LB_{k+1}, UB_{k+1})$ is primal feasible.

REMARK 16: Adding a nonbasic cell (e, f) to an existing basis B_k, obtaining a cycle Γ and dropping some $(r, s) \epsilon \Gamma$ is a standard step in primal algorithms for the (capacitated) transportation problem [6, 15] and hence no proof need be provided here for the well known result that

$$B_{k+1} = B_k - \{(r, s)\} + \{(e, f)\}$$

will be a basis if and only if $(r, s) \epsilon \Gamma$. Since (e, f) satisfies (80), the change from X_k to X_{k+1} corresponds to finding an alternate optimum [2, p. 49, 11]. If $(r, s) \neq (e, f)$ the new basis structure will have $B_{k+1} = B_k - \{(r, s)\} + \{(e, f)$ with appropriate changes in LB and UB. However, if $(r, s) = (e, f)$ the new basis structure will have $B_{k+1} = B_k$ with (e, f) moving LB_k to UB_{k+1} (if $x_{ef,k} = 0$) or from UB_k to LB_{k+1} (if $x_{ef,k} = U_{ef}$).

Among the adjacent bases $B_{k+1} = B_k - \{(r, s)\} + \{(e, f)\}$, Theorem 13 below determines the cell $(r, s) \epsilon \Gamma$ such that the solution X_{k+1} associated with $(B_{k+1}, LB_{k+1}, UB_{k+1})$ is primal feasible. To do this we let Γ, the cycle obtained by adding (e, f) to B_k, to be denoted by

(81) $$\Gamma = \{(e, f) = (i_0, j_0)\} \cup \{(i_1, j_1), (i_2, j_2) \;.\;.\;.\; (i_{2l-2}, j_{2l-2}), (i_{2l-1}, j_{2l-1})\}.$$

Then we define the sets of odd and even parity Γ_1 and Γ_2 (Definition3) as

(82) $$\Gamma_1 = \{(i_1, j_1), (i_3, j_3) \;.\;.\;.\; (i_{2l-1}, j_{2l-1})\}, \text{ and}$$

(83) $$\Gamma_2 = \{[(e, f) = (i_0, j_0)], (i_2, j_2) \;.\;.\;.\; (i_{2l-2}, j_{2l-2})\}.$$

THEOREM 13: Consider the solution X_k, and the sets Γ_1 and Γ_2 defined above. Find Δ by (i) or (ii) below depending on whether $x_{ef,k}=0$ or $=U_{ef}$.

(i) $x_{ef,k}=0$. Set

$$\Delta=\text{Min}\begin{cases} x_{ij,k} & \text{for } (i,j)\epsilon\Gamma_1 \\ (U_{ij}-x_{ij,k}) & \text{for } (i,j)\epsilon\Gamma_2. \end{cases} \tag{84}$$

(ii) $x_{ef,k}=U_{ef}$. Set

$$\Delta=-\text{Min}\begin{cases} (U_{ij}-x_{ij,k}) & \text{for } (i,j)\epsilon\Gamma_1 \\ x_{ij,k} & \text{for } (i,j)\epsilon\Gamma_2. \end{cases} \tag{85}$$

Then the solution given by (86) below is primal feasible.

$$x_{ij,k+1}=\begin{cases} x_{ij,k}-\Delta & \text{for } (i,j)\epsilon\Gamma_1 \\ x_{ij,k}+\Delta & \text{for } (i,j)\epsilon\Gamma_2 \\ x_{ij,k} & \text{elsewhere} \end{cases} \tag{86}$$

(a) If the minimum of (84) or (85) is attained at a cell $(r, s)\neq(e, f)$ then $B_{k+1}=B_k-\{(r, s)\}+\{(e, f)\}$ gives the adjacent basis with LB_k and UB_k appropriately modified to LB_{k+1} and UB_{k+1}.

(b) If the minimum of (84) or (85) is attained at (e, f) then the adjacent basis structure is given by $B_{k+1}=B_k$, with cell (e, f) changing from LB_k to UB_{k+1} (if $x_{ef,k}=0$) or from UB_k to LB_{k+1} (if $x_{ef,k}=U_{ef}$).

PROOF: All the proofs below are given for the case when $x_{ef,k}=0$. Similar proofs can be given when $x_{ef,k}=U_{ef}$.

By Remark 4 and from (86), it is clear that $\sum_{j\epsilon J} x_{ij,k+1}=\sum_{j\epsilon J} x_{ij,k}=a_i$ for $i\epsilon I$ and $\sum_{i\epsilon I} x_{ij,k+1}=\sum_{i\epsilon I} x_{ij,k}=b_j$ for $j\epsilon J$ so that conditions (4) and (5) are satisfied by the solution X_{k+1}. By the choice of Δ (see (84)) the conditions (6) are satisfied by all $x_{ij,k+1}$ for $i\epsilon I$ and $j\epsilon J$.

(a) We now have to show that this solution is basic (see Definition 4), i.e., $(i, j)\notin B_{k+1}\Rightarrow x_{ij}=0$ or U_{ij}. For the case when the minimum of (84) is taken at a cell $(r, s)\neq(e, f)$, the value of Δ is such that $x_{rs,k+1}=0$ or U_{rs} so that (r, s) satisfies the above condition. The other nonbasic cells for B_{k+1} also satisfy this condition since they are nonbasic for B_k as well and (86) does not alter the x_{ij} for these cells.

(b) When the minimum of (84) is taken on the cell (e, f), then by (84) and (86), $x_{ef,k+1}=U_{ef}$ so that (e, f) changes from LB_k to UB_{k+1} as claimed. The other nonbasic cells in B_{k+1} satisfy the condition that $(i, j)\notin B\Rightarrow x_{ij,k+1}=0$ or U_{ij} since they are nonbasic for B_k as well and (86) does not alter the x_{ij} for these cells.

The next two Lemmas examine the effect of the change in basis structure from (B_k, LB_k, UB_k) to $(B_{k+1}, LB_{k+1}, UB_{k+1})$ on the cost effects of the operators, i.e., on x_{pq}, $\sum_{i\epsilon I}\sum_{j\epsilon J}\gamma_{ij}x_{ij}$ (see Theorem 5).

LEMMA 5:

(a) For the operator δC^+_{pq}, $(p, q)\epsilon\Gamma_1$ if $x_{ef,k}=0$ and $(p, q)\epsilon\Gamma_2$ if $x_{ef,k}=U_{ef}$.

(b) For the operator δC^-_{pq}, $(p, q)\epsilon\Gamma_2$ if $x_{ef,k}=0$ and $(p, q)\epsilon\Gamma_1$ if $x_{ef,k}=U_{ef}$.

(c) For the operator δC_A, $\left(\sum_{(i,j)\epsilon\Gamma_2}\gamma_{ij}-\sum_{(i,j)\epsilon\Gamma_1}\gamma_{ij}\right)<0$ if $x_{ef,k}=0$. The amount

$$\left(\sum_{(i,j)\epsilon\Gamma_2}\gamma_{ij}-\sum_{(i,j)\epsilon\Gamma_1}\gamma_{ij}\right)>0 \text{ if } x_{ef,k}=U_{ef}.$$

PROOF:

(a) We prove below assuming $x_{ef,k}=0$ (i.e., $(e,f)\epsilon LB_k$). Let us first consider the case when $(p,q)\notin B_k$. From Step (2) of Algorithm 3, $(e,f)=(p,q)$ so that $(p,q)\epsilon LB_k$. From (31) it then follows that $\mu_k=M$ so that there was no necessity to change the basis. Consequently $(p,q)\epsilon B_k$. By (35), $(e,f)\epsilon[(I_p\times J_q)\cap LB]$. That is, $e\epsilon I_p$ and $f\epsilon J_q$. It then follows that the cycle Γ (81) should necessarily contain (p,q). By Remark 2 there is an even number of cells in the path connecting (e,f) and (p,q). Since $(e,f)\epsilon\Gamma_2$, $(p,q)\epsilon\Gamma_1$ as claimed. A similar proof applies when $x_{ef,k}=U_{ef}$. But in this case there is the additional possibility of $(p,q)=(e,f)$ which implies that $(p,q)\epsilon\Gamma_2$ (see (83)).

(b) The proof for this part is similar to that of (a).

(c) For the operator δC_A, the assumption $x_{ef,k}=0$ implies that $\gamma_{ef}-u_e^*-v_f^*<0$. (see (40)). By the well known relationship between the stepping stone method and the $u-v$ (MODI) method ([2], p. 531) $\gamma_{ef}-u_e^*-v_f^*=\sum_{(i,j)\epsilon\Gamma_2}\gamma_{ij}-\sum_{(i,j)\epsilon\Gamma_1}\gamma_{ij}$ so that Lemma 5(c) holds. A similar proof applies when $x_{ef,k}=U_{ef}$.

LEMMA 6:

(a) For the operator δC_{pq}^+, $x_{pq,k+1}\leqslant x_{pq,k}$.

(b) For the operator δC_{pq}^-, $x_{pq,k+1}\geqslant x_{pq,k}$.

(c) For the operator δC_A, $\sum_{i\epsilon I}\sum_{j\epsilon J}\gamma_{ij}x_{ij,k+1}\leqslant\sum_{i\epsilon I}\sum_{j\epsilon J}\gamma_{ij}x_{ij,k}$.

PROOF:

(a) Considering the case when $Y_{ef,k}=0$, from Lemma 5(a), $(p,q)\epsilon\Gamma_1$ so that from (86), $x_{pq,k+1}=x_{pq,k}-\Delta$. From (84), Δ is nonnegative since the solution X_k is primal feasible. Thus $x_{pq,k+1}\leqslant x_{pq}$. A similar proof applies when $x_{ef,k}=U_{ef}$.

(b) This proof is similar to that of (a).

(c) From (86) x_{ij} changes only on the cells of Γ so that

$$\sum_{i\epsilon I}\sum_{j\epsilon J}\gamma_{ij}\,(x_{ij,k+1}-x_{ij,k})=\Delta\left[\sum_{(i,j)\epsilon\Gamma_2}\gamma_{ij}-\sum_{(i,j)\epsilon\Gamma_1}\gamma_{ij}\right].$$

For the case when $x_{ef,k}=0$, $\Delta\geqslant 0$ from (84). Lemma 6(c) then directly follows from Lemma 5(c). A similar proof applies to the case when $x_{ef,k}=U_{ef}$ (in this case $\Delta\leqslant 0$ from (85)).

We summarize the main results of this section by the following Algorithm.

ALGORITHM 6: For applying the operator $\delta^*C(P)$.

(i) Let $k=1$, $P_1=P$, $B_1=B$, $LB_1=LB$, $UB_1=UB$, $X_1=X$, $Z_1=Z$, $u_{i,1}=u_i$ for $i\epsilon I$, $v_{j,1}=v_j$ for $j\epsilon J$.

(ii) Using Algorithm 3, determine μ_k and the entering cell (e,f). If $\sum_{i=1}^{k}\mu_i\geqslant\delta^*$, set $\delta'=\delta^*$, $t'=k$ and go to (iv). Otherwise apply $\mu_k\ C\ (P_k)$ using Algorithm 4. Define the resulting problem P_{k+1} with

optimal dual solutions $u_{i,k+1}$ for $i \epsilon I$ and $v_{j,k+1}$ for $j \epsilon J$ and cost Z_{k+1}. (The primal solution X_k does not get altered by this transformation).

(iii) Determine the cycle Γ obtained by adding (e, f) to B_k and identify Γ_1 and Γ_2 as per (82)–(83). Use (84) or (85) depending on whether $x_{ef,k}=0$ or U_{ef} and determine the amount Δ. If the minimum is taken on a cell $(r, s) \epsilon B_k$ then define $B_{k+1}=B_k-\{(r, s)\}+\{(e, f)\}$ and adjust the sets LB_k and UB_k accordingly to obtain LB_{k+1} and UB_{k+1}. If the minimum is taken on the cell (e, f) itself, move (e, f) from LB_k to UB_{k+1} if $x_{ef,k}=0$, (move (e, f) from UB_k to LB_{k+1} if $x_{ef,k}=U_{ef}$) and define $B_{k+1}=B_k$. Obtain X_{k+1} using (86). Set $k=k+1$ and go to (ii).

(iv) To obtain the optimal solution for a problem P^T defined by the operator $\delta\mathbf{C}(P)$, $(0 \leqslant \delta \leqslant \delta^*)$, obtain t and λ from (51) and (52) and apply $\lambda C\ (P_t)$ using Algorithm 4. STOP.

REMARK 17: We now derive an expression for the optimal cost Z as a function δ in the range $0 \leqslant \delta \leqslant \delta^*$. By (49)–(52) and Algorithm 4, it follows that

$$Z^+\ (\delta)=Z_1+\sum_{k=1}^{t-1} \mu_k x_{pq,k}+\lambda x_{pq,t} \qquad \text{for the operator } \delta\mathbf{C}^+_{pq},$$

$$Z^-\ (\delta)=Z_1-\sum_{k=1}^{t-1} \mu_k x_{pq,k}-\lambda x_{pq,t} \qquad \text{for the operator } \delta\mathbf{C}^-_{pq}, \text{ and}$$

$$Z^A\ (\delta)=Z_1+\left[\sum_{k=1}^{t-1} \mu_k \left(\sum_{i\epsilon I}\sum_{j\epsilon J} \gamma_{ij} x_{ij,k}\right)\right]+\lambda \sum_{i\epsilon I}\sum_{j\epsilon J} \gamma_{ij} x_{ij,t} \qquad \text{for the operator } \delta\mathbf{C}_A,$$

where t and λ are as defined by (51)–(52). The above expressions show that the cost curves are piecewise linear. From Lemma 6 and from the above equations, the cost functions $Z\ (\delta)$ can be recognized to be concave, a well known result in parametric programming (of the "price vector") for general linear programming problems.

EXAMPLE (cont'd): We illustrate with Fig. 4 the application of the same cost operator $\delta\mathbf{C}^+_{pq}\ (P)$ discussed in section 5 of [20] with $(p, q)=(1, 1)$. Fig. 1(a) shows the problem $P_1=P$ with optimal solutions X_1, $u_{i,1}$ for $i \epsilon I$ and $v_{j,1}$ for $j \epsilon J$ with basis structure (B_1, LB_1, UB_1) and optimal cost $Z_1=945$. In section 5 of [20] it was shown that the basis preserving operator $\delta C^+_{11}\ (P_1)$ can be applied to a maximum extent of $\mu_1=3$ with $(e, f)=(2, 4)$ as the entering cell. Assuming that we are interested in the optimum solution for $0 \leqslant \delta \leqslant \delta^*=1{,}000$ (say), at step (ii) of Algorithm 6 we find $\sum_{i=1}^{1} \mu_1 < \delta^*$ so that we apply $3C^+_{11}\ (P_1)$ to obtain problem P_2 with optimal dual solutions $u_{i,2}$ and $v_{j,2}$ as shown in Fig. 2(b) with cost $Z_2=Z_1+\mu_1\ x_{11,1}=945+75=1{,}020$. We note that the cell $(e, f)=(2, 4)$ satisfies (80). The optimum dual solution for $0 \leqslant \delta \leqslant 3$ is shown in Fig. 2(a) with a cost of $Z=945+25\delta$. We note that the primal solution X_1 remains unchanged during this transformation.

In step (iii) of Algorithm 6 we find the cycle Γ obtained by adding $(e, f)=(2, 4)$ to the basis B_1. This cycle is shown in Fig. 4(a) with $\Gamma_1=\{(3, 4), (1, 1), (2, 3)\}$, and $\Gamma_2=\{(2, 4), (3, 1), (1, 3)\}$. Since $x_{ef,1}=0$ we apply (84) and obtain $\Delta=5$ with $(r, s)=(1, 3)$. We now transform the solution X_1 as per (86) to obtain the solution X_2 shown in Fig. 4(b) with $B_2=B_1-\{(1, 3)\}+\{(2, 4)\}$. The reader may verify that this solution is primal feasible, i.e., satisfies (4)–(6). Also $UB_2=UB_1+\{(1, 3)\}$ and $LB_2=LB_1-\{(2, 4)\}$. Note that Lemmas 5(a) and 6(a) hold true. The cost effect has dropped from $x_{11,1}=25$ to $x_{11,2}=20$.

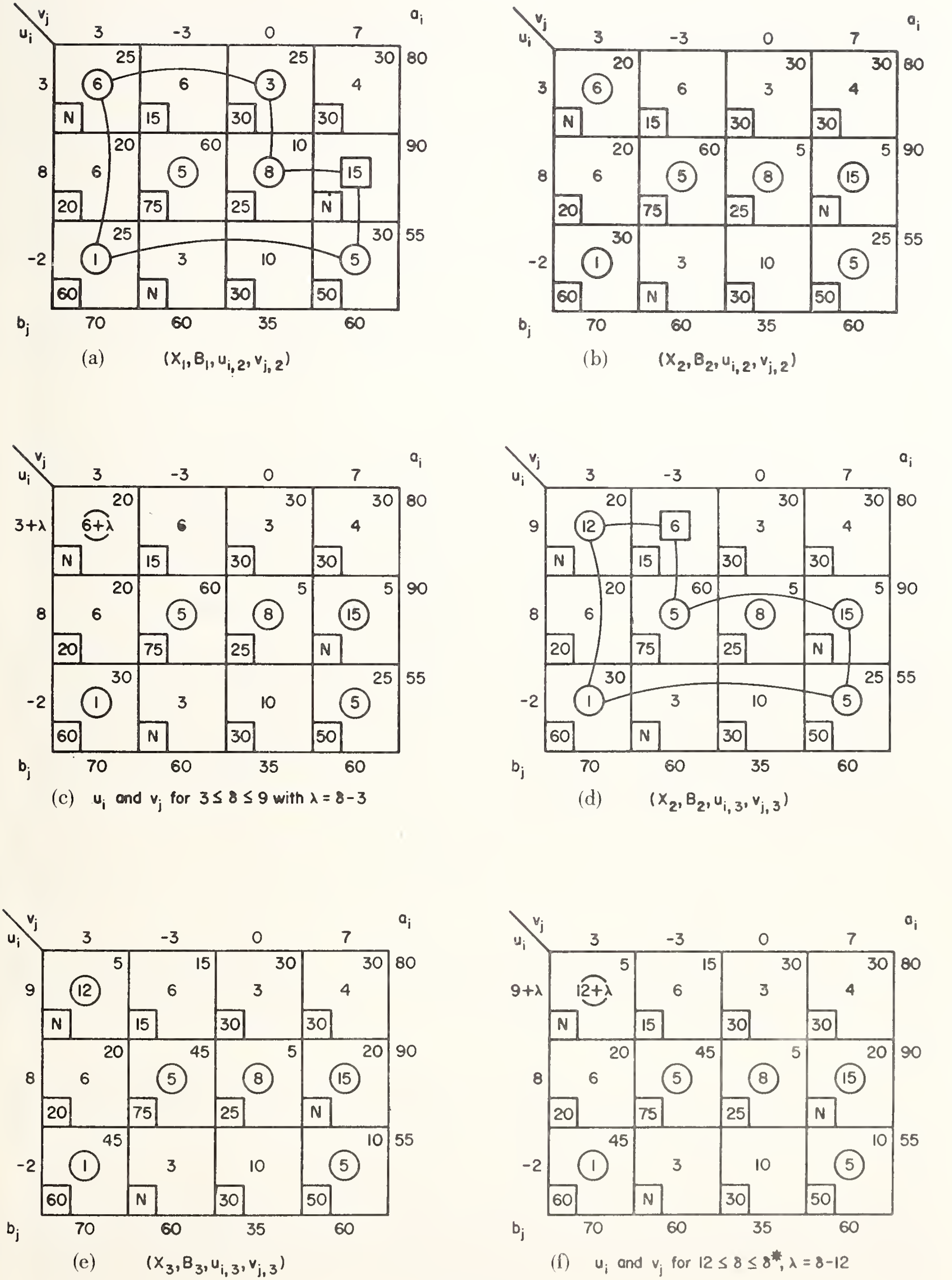

(a) $(X_1, B_1, u_{i,2}, v_{j,2})$
(b) $(X_2, B_2, u_{i,2}, v_{j,2})$
(c) u_i and v_j for $3 \leq \delta \leq 9$ with $\lambda = \delta - 3$
(d) $(X_2, B_2, u_{i,3}, v_{j,3})$
(e) $(X_3, B_3, u_{i,3}, v_{j,3})$
(f) u_i and v_j for $12 \leq \delta \leq \delta^*$, $\lambda = \delta - 12$

FIGURE 4

Returning to step (ii) of Algorithm 6, we determine μ_2 associated with the basis in Fig. 4(b) using Algorithm 3. In step (3) of Algorithm 3, we find $I_p=\{1\}$, $J_p=\phi$, $I_q=\{2,3\}$ and $J_q=\{1, 2, 3, 4\}$. From (35) $\mu_2=6$ with $(e, f)=(1, 2)$. Again $\sum_{i=1}^{2} \mu_i=9<\delta^*$ so that we apply $6C_{11}^{+}$ (P_2) to obtain P_3 with dual solutions $u_{i,3}$ and $v_{j,3}$ as shown in Fig. 4(d) and cost $Z_3=Z_2+\mu_2 x_{11,2}=1{,}020+120=1{,}140$. The optimum solution for other values of $3\leqslant\delta\leqslant 9$ is shown in Fig. 4(c) with $\lambda=\delta-3$ and cost $Z=Z_2+20\lambda$. In step (iii) of Algorithm 6, we add (1, 2) to B_2 to obtain the cycle shown in Fig. 4(d) with $\Gamma_1=\{(2, 2), (3, 4), (1, 1)\}$ and $\Gamma_2=\{(1, 2), (2, 4), (3, 1)\}$. Since $x_{ef,2}=0$, we use (84) and obtain $\Delta=15$ with $(r, s)=(e, f)=(1, 2)$. Thus $B_3=B_2$, and the cell (1, 2) changes from LB_2 to UB_3. The solution X_3 obtained using (86) is shown in Fig. 4(e). The reader may again verify Lemmas 5(a) and 6(a). (The cost effect has now dropped from 20 to 5).

We now return to step (ii) and determine μ_3 using Algorithm 3. The sets I_p, I_q, J_p, and J_q remain the same as before so that $I_p\times J_q-\{(p, q)\}$ $\{(1, 2), (1, 3), (1, 4)\}$. Since all these cells are in UB_3, the sets over which the minimization (35) is done is empty so that $\mu_3=M$. Consequently $\sum_{i=1}^{3} \mu_i>\delta^*$. The optimum solution for $9\leqslant\delta\leqslant\delta^*$ is shown in Fig. 4(f). We note that $x_{11,3}=5$ regardless of however large c_{11} becomes. From row 1 it can be seen that other routes from row 1 are already used to the maximum extent so that there is no feasible solution to this problem with $x_{11}<5$ (Remark 11). The cost curve $Z^+(\delta)$ in this example is $\{[945+25\delta]$ for $0\leqslant\delta\leqslant 3$, $[1{,}020+20\ (\delta-3)]$ for $3\leqslant\delta\leqslant 9$, and $[1{,}140+5\ (\delta-9)]$ for $\delta\geqslant 9\}$ verifying Remark 17 that the cost curve is concave.

We now consider the Fig. 5 the area cost operator $\delta\mathbf{C}_A$ with γ_{ij} shown in Fig. 2(c). In section 5 of [20] we showed that $\mu_1=1$ with $(e, f)=(1, 2)$. Figure 2(d) gives the problem P_2 and the dual solutions $u_{i,2}$ and $v_{j,2}$. The optimum solution u_i and v_j for any $0\leqslant\delta\leqslant 1$ is given by (39) with $u_i=u_{i,1}$ and $v_j=v_{j,1}$ (Fig.1(a)) and u_i^* and v_j^* as shown in Fig. 2(c) with optimal cost $Z=Z_1+\delta\sum_{i\in I}\sum_{j\in J}\gamma_{ij}x_{ij}=945+860\delta$.

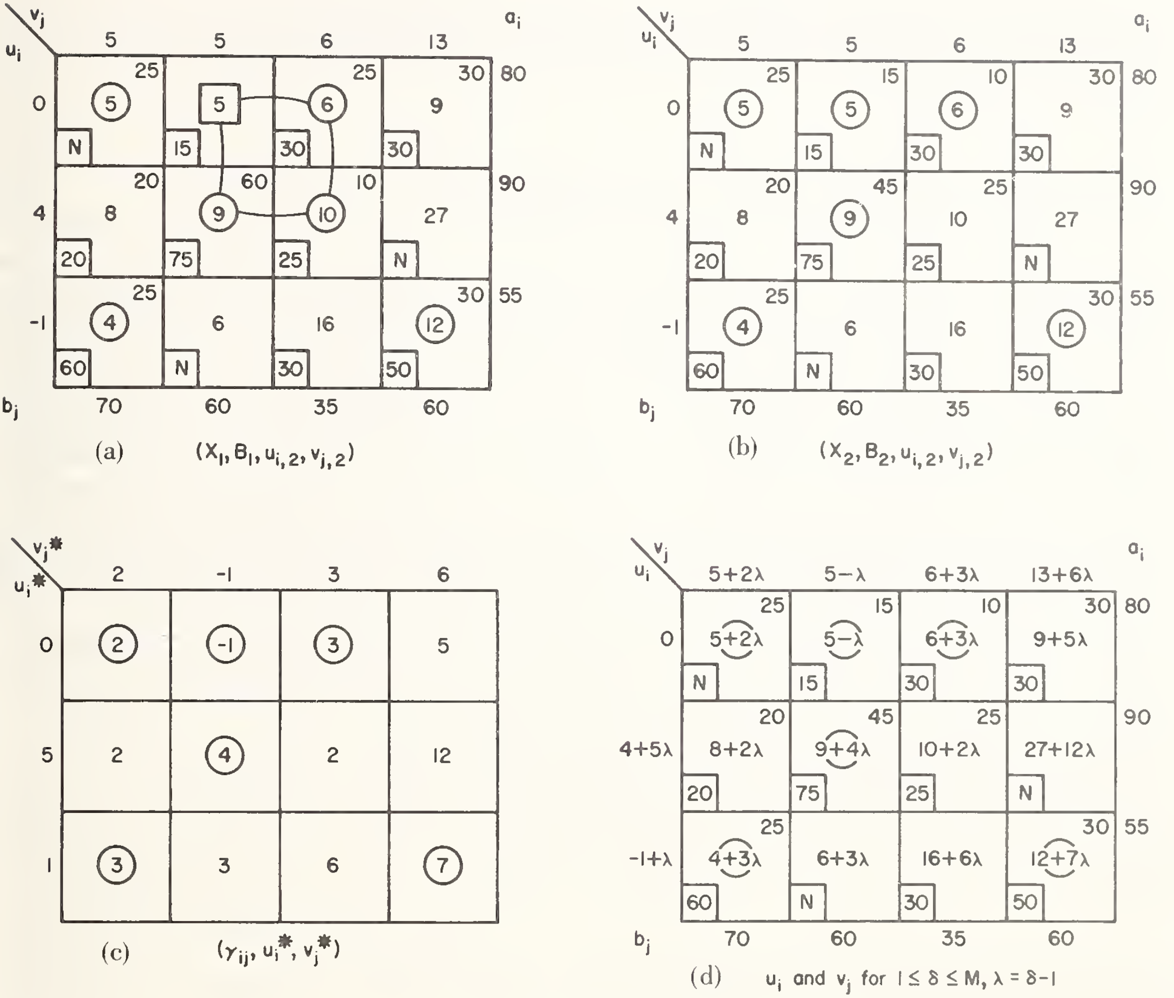
Adding (1, 2) to B_1 we obtain the cycle shown in Fig. 5(a), where $\Gamma_1=\{(1, 3), (2, 2)\}$ and $\Gamma_2=\{(1, 2), (2, 3)\}$. Since $x_{ef,1}=0$ we use (84) and obtain $(r, s)=(2, 3)$ and $\Delta=15$. Thus $B_2=B_1-\{(2, 3)\}+\{(1, 1)\}$ as shown in Fig. 5(b) with X_2 determined from (86). Lemma 5(c) is verified in this case since $\sum_{(i,j)\in\Gamma_2}\gamma_{ij}=\gamma_{12}+\gamma_{23}=1$ and $\sum_{(i,j)\in\Gamma_1}\gamma_{ij}=7$. The cost effect $\sum_{i\in I}\sum_{j\in J}\gamma_{ij}x_{ij,2}=770$ and is less than 860 in the previous iteration thus verifying Lemma 6(c). Fig. 5(c) shows the u_i^*, v_j^* associated with basis B_2. Since the relevant sets over which the minimization (40) is done, are empty, $\mu_2=M$. Fig. 5(d), shows the optimum solution for all values of $\delta\geqslant 1$. The reader may verify that this solution is both primal and dual feasible.

To prove convergence of Algorithm 6, we need to assume that the problem is primal nondegenerate.

DEFINITION 11: The problem P is *primal nondegenerate* if for any basis structure (B, LB, UB), the associated primal solution X (satisfying (4) and (5)) satisfies the condition:

$$(i,j)\in B\Rightarrow(x_{ij}\neq 0\quad\text{and } x_{ij}\neq U_{ij}). \tag{87}$$

REMARK 18: Let us assume, without loss of generality, that a_i, b_j, and U_{ij} are scaled (if necessary) to be integers and that $U_{ij}\geqslant 1$ for all i and j. Then the standard degeneracy prevention scheme [6, 11] given below would ensure that condition (87) is satisfied, we set



(a) $(X_1, B_1, u_{i,2}, v_{j,2})$

(b) $(X_2, B_2, u_{i,2}, v_{j,2})$

(c) $(\gamma_{ij}, u_i^*, v_j^*)$

(d) u_i and v_j for $1 \le \delta \le M$, $\lambda = \delta - 1$

FIGURE 5

(88) $$a_i' = a_i + \epsilon \qquad \text{for } i \epsilon I,$$

(89) $$b_j' = b_j \qquad \text{for } j \epsilon J - \{n\} \text{ with } b_n' = b_n + m\epsilon, \text{ where}$$

(90) $$0 < \epsilon < 1/(m+1),$$

and use the rim conditions $\{a_i'\}$, $\{b_j'\}$ instead of $\{a_i\}$ and $\{b_j\}$.

THEOREM 14: Assuming that the problem P is primal nondegenerate, Algorithm 6 terminates in a finite number of steps.

PROOF: By the assumption (87), the value of Δ obtained in (84) or (85) is nonzero (recall that by assumption, $U_{ef} \geq 1$). From the proofs of Lemmas 6(a)–(c), it is easy to verify that the inequalities in the statements of Lemma 6(a)–(c) hold as strict inequalities. Thus the value of x_{pq} is strictly decreasing (increasing) for the δC_{pq}^+ (δC_{pq}^-) operator. Similarly $\sum_{i \epsilon I} \sum_{j \epsilon J} \gamma_{ij} x_{ij}$ is strictly decreasing for the δC_A operator.

These values are unique for any given basis structure (B, LB, UB) so that no basis structure will be repeated. The total number of possible basis structures is finite so that Algorithm 6 terminates in a finite number of steps.

5. ECONOMIC AND MANAGERIAL INTERPRETATIONS OF OPERATORS

In this section we make use of the operator theory in determining the effects on the optimal cost of *ceteris paribus* increase (or decrease) in the capacity of a warehouse or the requirement of a market. Such information is useful to a manager in making decisions as to which warehouse capacities are to be increased, which markets should be sought after, etc. We also determine the *downward marginal cost*, i.e., the rate at which the optimum cost would be reduced if the total physical volume handled in the transportation system were reduced. In a similar manner, examining the effects of an increase or decrease in the unit costs of transportation would be of help in bargaining rates with trucking companies and in reacting to changed conditions caused by rate changes, strikes, etc. It should be noted that though the formulation of the problem (3)–(6) is based on transportation costs alone, by defining c_{ij} to be the negative of the unit profit of supplying from warehouse (or factory) i to market j, the transportation problem (3)–(6) becomes a more general profit maximization model.

In the study of such problems it becomes necessary to relax assumption (A3). For instance the total amount of warehouse supplies, $\sum_{i\epsilon I} a_i$, may be greater than total market demands, $\sum_{j\epsilon J} b_j$, or conversely. We define an augmented problem P' by defining the index sets

$$I' = I \cup \{m+1\}, \tag{91}$$

$$J' = J \cup \{n+1\}. \tag{92}$$

The $m+1$st warehouse can be considered a dummy or slack warehouse. Similarly $(n+1)$ is a dummy market. We further define.

$$c_{i,n+1} = 0, \qquad U_{i,n+1} = N \qquad \text{for } i\epsilon I, \tag{93}$$

$$c_{m+1,j} = 0, \qquad U_{m+1,j} = N \qquad \text{for } j\epsilon J, \tag{94}$$

$$c_{m+1,n+1} = M, \qquad U_{m+1,n+1} = N, \tag{95}$$

$$a_{m+1} = 0 \qquad \text{and } b_{n+1} = \sum_{i\epsilon I} a_i - \sum_{j\epsilon J} b_j \qquad \text{if} \sum_{i\epsilon I} a_i \geq \sum_{j\epsilon J} b_j, \tag{96}$$

$$a_{m+1} = \sum_{j\epsilon J} b_j - \sum_{i\epsilon I} a_i \qquad \text{and } b_{n+1} = 0 \qquad \text{if} \sum_{j\epsilon J} b_j \geq \sum_{i\epsilon I} a_i. \tag{97}$$

Then we define the augmented problem P' as (3)–(6) with the index sets I and J replaced by I' and J'. Since $x_{i,n+1} \geq 0$ this implies that $\sum_{j\epsilon J} x_{ij} \leq a_i$ for $i\epsilon I$. Similarly $\sum_{i\epsilon I} x_{ij} \leq b_j$ for $j\epsilon J$. Since I and J denote the real warehouses and markets, the *total physical volume* K handled in the system is given by

(98) $$K=\sum_{i\epsilon I}\sum_{j\epsilon J} x_{ij}=\sum_{i\epsilon I} a_i-\sum_{i\epsilon I} x_{i,n+1}=\sum_{j\epsilon J} b_j-\sum_{j\epsilon J} x_{m+1,j}.$$

The reader may verify that $a_{m+1}\left(=\sum_{j\epsilon J} x_{m+1,j}\right)$ and $b_{n+1}\left(=\sum_{i\epsilon I} x_{i,n+1}\right)$ are defined so that

(99) $$K=\mathrm{Min}\left(\sum_{i\epsilon I} a_i, \sum_{j\epsilon J} b_j\right).$$

The above formulation of the augmented problem is somewhat different from those in [6, 11, 15]. For instance, when $\sum_{i\epsilon I} a_i>\sum_{j\epsilon J} b_j$ those references would add a dummy warehouse, but no dummy market. For some of the analysis below, it is necessary to include both a dummy warehouse and a dummy market.

As a direct consequence of Theorem 1, it follows that d_{ij} $(-d_{ij})$ gives the shadow price, i.e., the rate at which the optimal cost increases, if capacity of warehouse i and requirement of market j were simultaneously increased (decreased). As shown in Fig. 1(b), d_{ij} could be negative even if all c_{ij} were positive. Similarly by Theorem 5, x_{ij} $(-x_{ij})$ gives the shadow price when the unit cost c_{ij} is increased (decreased). By (6), $x_{ij}\geqslant 0$.

We now consider the case when $\sum_{i\epsilon I} a_i\geqslant\sum_{j\epsilon J} b_j$ and let us say that the capacity of the pth warehouse is increased from a_p to $a_p+\delta$ *ceteris paribus* (i.e., none of the other warehouse capacities a_i for $i\epsilon I$ with $i\neq p$ or market requirements b_j for $j\epsilon J$ are changed). From (96) it follows that for the augmented problem the only other change involved is the increase of b_{n+1} to $b_{n+1}+\delta$. Consequently $\delta\mathbf{R}^+_{p,n+1}$ investigates the effect of *ceteris paribus* increase in a_p. By Theorem 1 the shadow price is given by $d_{p,n+1}$. By (95) the upper bound on the variable $x_{p,n+1}$ is chosen to be so large that $(p, n+1)\notin UB$. Thus from (12), (13), and (93) it follows that

$$d_{p,n+1}=u_p+v_{n+1}\leqslant c_{p,n+1}=0,$$

so that this shadow price is nonpositive. By (99) the total physical volume K remains equal to $\sum_{j\epsilon J} b_j$ despite the increase in a_p since we assumed $\sum_{i\epsilon I} a_i\geqslant\sum_{j\epsilon J} b_j$. The increase in a_p thus increases the flexibility and consequently cannot increase the optimal cost.

We now investigate the effect of increasing the qth market requirement from b_q to $b_q+\delta$ *ceteris paribus*, under the assumption $\sum_{i\epsilon I} a_i>\sum_{j\epsilon J} b_j$ so that from (96), $b_{n+1}>0$ for the augmented problem. We assume the amount δ is such that $\sum_{i\epsilon I} a_i\geqslant\sum_{j\epsilon J} b_j+\delta$. From (99), K increases to $K+\delta$ and from (96), b_{n+1} decreases to $b_{n+1}-\delta$. Since $b_{n+1}>0$, there must be a basis cell $(k, n+1)$ such that $x_{k,n+1}>0$. We find that an increase of b_q to $b_q+\delta$ and a simultaneous decrease of b_{n+1} to $b_{n+1}-\delta$ is equivalent to a simultaneous application of $\delta\mathbf{R}^+_{k,q}$ and $\delta\mathbf{R}^-_{k,n+1}$. (From Theorem 2 the application of $\delta\mathbf{R}^-_{k,n+1}$ merely consists of decreasing $x_{k,n+1}$ to $x_{k,n+1}-\delta$; consequently δ should be less than $x_{k,n+1}$—if it is more we will have to choose other basic cell(s) $(k', n+1)$ in addition.) Since $(k, n+1)\epsilon B$, by Theorem 1 and from (93) the cost effect $d_{k,n+1}=c_{k,n+1}=0$ so that the cost effect of a *ceteris paribus* increase in b_q is given by d_{kq}.

TABLE 2. *Economic Interpretations of Operators*

Data for the transformed problem same as the original problem *except*	Operator and cost effect (shadow price)[a]		
	$\sum_{i\in I} a_i > \sum_{j\in J} b_j$	$\sum_{i\in I} a_i = \sum_{j\in J} b_j$	$\sum_{i\in I} a_i < \sum_{j\in J} b_j$
$a_p^{\pm} = a_p \pm \delta$ $b_q^{\pm} = b_q \pm \delta$	$\delta\mathbf{R}_{pq}^{\pm}$ $(\pm d_{pq})$	$\delta\mathbf{R}_{pq}^{\pm}$ $(\pm d_{pq})$	$\delta\mathbf{R}_{pq}^{\pm}$ $(\pm d_{pq})$
$a_p^{+} = a_p + \delta$	$\delta\mathbf{R}_{p,n+1}^{+}$ $(d_{p,n+1})$	$\delta\mathbf{R}_{p,n+1}^{+}$ $(d_{p,n+1})$	$\delta\mathbf{R}_{pl}^{+}[\delta\mathbf{R}_{m+1,l}^{-}]$ (d_{pl})
$a_p^{-} = a_p - \delta$	$\delta\mathbf{R}_{p,n+1}^{-}$ $(-d_{p,n+1})$	$\delta\mathbf{R}_{pl}^{-}[\delta\mathbf{R}_{m+1,l}^{+}]$ $(-d_{pl})$	$\delta\mathbf{R}_{pl}^{-}[\delta\mathbf{R}_{m+1,l}^{+}]$ $(-d_{pl})$
$b_q^{+} = b_q + \delta$	$\delta\mathbf{R}_{kq}^{+}[\delta\mathbf{R}_{k,n+1}^{-}]$ (d_{kq})	$\delta\mathbf{R}_{m+1,q}^{+}$ $(d_{m+1,q})$	$\delta\mathbf{R}_{m+1,q}^{+}$ $(d_{m+1,q})$
$b_q^{-} = b_q - \delta$	$\delta\mathbf{R}_{kq}^{-}[\delta\mathbf{R}_{k,n+1}^{+}]$ $(-d_{kq})$	$\delta\mathbf{R}_{kq}^{-}[\delta\mathbf{R}_{k,n+1}^{+}]$ $(-d_{kq})$	$\delta\mathbf{R}_{m+1,q}^{-}$ $(-d_{m+1,q})$
$K^{-} = K - \delta$	$\delta\mathbf{R}_{m+1,n+1}^{+}$ $(d_{m+1,n+1})$	$\delta\mathbf{R}_{m+1,n+1}^{+}$ $(d_{m+1,n+1})$	$\delta\mathbf{R}_{m+1,n+1}^{+}$ $(d_{m+1,n+1})$
$c_{pq}^{\pm} = c_{pq} \pm \delta$	$\delta\mathbf{C}_{pq}^{\pm}$ $(\pm x_{pq})$	$\delta\mathbf{C}_{pq}^{\pm}$ $(\pm x_{pq})$	$\delta\mathbf{C}_{pq}^{\pm}$ $(\pm x_{pq})$

[a] Expressions in parentheses denote the cost effects (shadow prices). The indices k and l are such that $(k, n+1)$ and $(m+1, l)$ are in the current basis.

We now consider "downward marginal cost"—the rate at which the optimum cost decreases if the total volume handled in the system decreases, i.e., K is decreased to $K - \delta$. From (98) this means that $\sum_{i\in I} x_{i,n+1}$ and $\sum_{j\in J} x_{m+1,j}$ should each increase by δ, i.e., a simultaneous increase of b_{n+1} to $b_{n+1} + \delta$ and a_{m+1} to $a_{m+1} + \delta$. Consequently $\delta\mathbf{R}_{m+1,n+1}^{+}$ is the relevant operator with a cost effect of $d_{m+1,n+1}$.' Since "downward marginal cost" is defined as a decrease in cost it is equal to $-d_{m+1,n+1}$. Similarly an analysis of changing K to $K + \delta$ would mean application of $\delta\mathbf{R}_{m+1,n+1}^{-}$ which is not possible since we have defined a_{m+1} and b_{n+1} such that at least one of them is zero. However if we define upward marginal cost as the rate at which cost would increase if K is increased to $K + \delta$ subject to *lower bounds* on the amounts supplied from each warehouse and to each market (the current formulation (3)–(6) requires $\sum_{j\in J} x_{ij} \leqslant a_i$ since $x_{i,n+1} \geqslant 0$), the formulation in [18] can be used for determining the upward marginal cost.

The cost effects of other possibilities of increasing or decreasing the capacity of a warehouse or the requirement of a market (under different assumptions on the relationship of $\sum_{i\in I} a_i$ with respect to $\sum_{j\in J} b_j$) can be similarly derived. These results are displayed in Table 2.

The duals d_{ij}, $d_{m+1,n+1}$ etc., may not yield the "real shadow price," i.e., the rate at which the optimal cost would change if a nonzero change is made in the data of the problem. To obtain the real

shadow price we will have to apply the relevant operators over $\delta > 0$. This may involve altering the basis structure using Algorithms 5 (or 6) until a positive range μ is obtained. The relevant operators could, of course, be used to study such *ceteris paribus* effects over larger ranges if desired—with the caution that the operators to be used may be quite different depending on the 'current' values of $\sum_{i\epsilon I} a_i$ and $\sum_{j\epsilon J} b_j$. For instance, referring to table 2 we note that if the market requirement b_q has to be increased *ceteris paribus* for a problem with $\sum_{i\epsilon I} a_i > \sum_{j\epsilon J} b_j$, we apply $\delta\mathbf{R}^+_{kq}[\delta\mathbf{R}^-_{k,n+1}]$ for $0 \leqslant \delta \leqslant \sum_{i\epsilon I} a_i - \sum_{j\epsilon J} b_j$. Beyond this range $\sum_{i\epsilon I} a_i \leqslant \sum_{j\epsilon J} b_j$ so that we have to apply $\delta\mathbf{R}^+_{m+1,q}$.

In this paper we were primarily interested in parametric programming. In section 2 showed how a post-optimization problem (e.g., obtaining the optimal solution for data $\hat{a}_i$, $\hat{b}_j$ (say) starting with the optimal solution for data a_i, b_j) can be converted to a parametric program by suitably defining an area operator (instead we could use a sequence of cell operators). A method for solving capacitated transportation problem would then be to post optimize a problem in which all the a_i's and b_j's are zero (and for which it is easy to construct an optimum solution) to the problem where the a_i's and b_j's take on the required values [1, 12]. Similarly post optimizing a problem for which all c_{ij}'s are zero (for which an optimum solution can be easily found) provides yet another method of solving transportation problems. The computational characteristics of some of these algorithms are presently being explored by the authors.

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QUADRATIC PROGRAMMING WITH QUADRATIC CONSTRAINTS

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ABSTRACT

A program with a quadratic objective function and quadratic constraints is considered. Two duals to such programs are provided, and an algorithm is presented based upon approximations to the duals. The algorithm consists of a sequence of linear programs and programs involving the optimization of a quadratic function either unconstrained or constrained to the nonnegative orthant. An example involving production planning is presented.

I. INTRODUCTION

Programs with a quadratic objective function and quadratic constraints were first introduced by Kuhn and Tucker in their classic paper [4]. They showed that given certain conditions the solution is equivalent to the solution of a saddle point problem. This paper presents duality results for quadratic programs with quadratic constraints and develops an algorithm for the solution of such programs based upon the approximation of the dual. The algorithm consists of a sequence of linear programs and programs involving the optimization of a quadratic function either unconstrained or constrained to the nonnegative orthant. An example is presented involving production planning.

Duality results for quadratically constrained quadratic programs have been obtained by Peterson and Ecker [7], [8], [9], from the point of view of geometric inequalities. The duality development in this paper is in the spirit of Geoffrion [5]. Sinha [11] has considered programs with a single quadratic constraint, and van de Panne [12] has developed an algorithm for such programs which converges in a finite number of iterations.

II. DUALITY

Consider the following primal program:

$$(P) \qquad z = \text{maximize } c'x + x'Vx$$
$$\text{subject to } a_i x + x'W_i x \leqslant b_i; \qquad i = 1, \ldots, m,$$
$$x \geqslant 0,$$

where c and x are n-component column vectors, V and W_i, $i = 1, \ldots, m$, are symmetric $n \times n$ matrices, b is an m-component vector, a_i, $i = 1, \ldots, m$, are n-component row vectors with

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix},$$

and (′) denotes transpose. Two equivalent programs dual to (P) are

$$(D_1) \qquad z_1 = \underset{\lambda \geq 0}{\text{minimize}} \; [\underset{x \geq 0}{\text{supremum}} \; L_1(x, \lambda)],$$

where $L_1(x, \lambda) = c'x + x'Vx + \sum_{i=1}^{m} \lambda_i(b_i - a_ix - x'W_ix)$, and

$$(D_2) \qquad z_2 = \underset{\substack{\lambda \geq 0 \\ \gamma \geq 0}}{\text{minimize}} \; [\underset{x \in E^n}{\text{supremum}} \; L_2(x, \lambda. \gamma)],$$

where $L_2(x, \lambda, \gamma) = L_1(x, \lambda) + \gamma'x$ and E^n denotes Euclidean n-space.

The algorithm to be presented in the next section utilizes approximations to the computational forms of (D_1) and (D_2) to obtain a solution to (P). As an introduction to the theory on which the algorithm is based, the duality theorems relating (P) with (D_1) and (D_2) will be presented. First a computational form of (D_2) will be given. Assuming that $L_2(x, \lambda, \gamma)$ has a finite maximizer x for any (λ, γ), that maximizer must satisfy

$$(1) \qquad c + 2Vx - A'\lambda - 2\sum_{i=1}^{m} \lambda_i W_ix + \gamma = 0.$$

(D_2) is thus equivalent to

$$(D_2^*) \qquad z_2 = \underset{x, \lambda, \gamma}{\text{minimize}} \; [-x'Vx + \sum_{i=1}^{m} \lambda_i x'W_ix + \lambda'b]$$

$$\text{subject to } c + 2Vx - A'\lambda - 2\sum_{i=1}^{m} \lambda_i W_ix + \gamma = 0$$

$$\lambda \geq 0, \qquad \gamma \geq 0$$

where the objective function in (D_2^*) is obtained by postmultiplying (1) transposed by x and subtracting that from $L_2(x, \lambda, \gamma)$. The dual program (D_1) restricts x to be in the nonnegative orthant. Theorems 1 and 2 below indicate conditions under which the constraints $x \geq 0$ may be added to the following dual program (analogous to (D_1)) without affecting the solution.

$$(D_1^*) \qquad z_1 = \underset{x, \lambda}{\text{minimize}} \; [-x'Vx + \sum_{i=1}^{m} \lambda_i x'W_ix + \lambda'b]$$

$$\text{subject to } c + 2Vx - A'\lambda - 2\sum \lambda_i W_ix \leq 0$$

$$\lambda \geq 0.$$

The following theorem indicates that (D_1^*) and (D_2^*) are duals of (P) and that solving (D_2^*) solves (D_1^*) and vice versa.

THEOREM 1: Let V be negative semidefinite and W_i, $i = 1, \ldots, m$, be positive semidefinite. If $\hat{x}$ solves (P) and a constraint qualification* is satisfied at $\hat{x}$, there exists $\hat{\lambda} \geq 0$ and $\hat{\gamma} \geq 0$ such that $(\hat{x}, \hat{\lambda}, \hat{\gamma})$ solves (D_2^*) (and equivalently (D_2)), $(\hat{x}, \hat{\lambda})$ solves (D_1^*) (and equivalently (D_1)), and $z = z_1 = z_2$. Conversely, if $(\hat{x}, \hat{\lambda})$ solves (D_1^*), there exists $\hat{\gamma} \geq 0$ such that $(\hat{x}, \hat{\lambda}, \hat{\gamma})$ solves (D_2^*).

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*For the theorem any of the usual constraint qualifications may be used (see [6]), but since it will be used in the convergence proof, assume that there exists an x^0 such that $a_ix^0 + x^{0\prime}W_ix^0 < b_i$ for all i.

PROOF: The existence of $\hat{\lambda} \geq 0$ and $\hat{\gamma} \geq 0$ follows from the Kuhn-Tucker Theorem. That $(\hat{x}, \hat{\lambda}, \hat{\gamma})$ solves (D_2^*) and that $z = z_2$ follows from Geoffrion's Theorem 3 [5], since the objective function of (P) is concave and the constraints convex. To show that $(\hat{x}, \hat{\lambda})$ solves (D_1^*), note that $\hat{x} \geq 0$ from (P), and $\hat{\gamma} \geq 0$ implies that $(\hat{x}, \hat{\lambda})$ is feasible for (D_1^*). Since the objective functions of (D_1^*) and (D_2^*) are the same, and every (x, λ) satisfying the constraints of (D_1^*) is feasible for (D_2^*), $(\hat{x}, \hat{\lambda})$ solves (D_1^*) and $z_2 = z_1$. Conversely, if $(\hat{x}, \hat{\lambda})$ solves (D_1^*), there exists $\hat{\gamma} \geq 0$ such that $(\hat{x}, \hat{\lambda}, \hat{\gamma})$ solves (D_2^*), since if (x, λ, γ) is feasible for (D_2^*), then (x, λ) is feasible for (D_1^*). Q.E.D.

The next theorem presents conditions under which a solution to a dual (D_1^*) or (D_2^*) yields a solution to the primal.

THEOREM 2: If either (a) V is negative definite, or (b) some $\bar{\lambda}_i > 0$ with corresponding W_i positive definite, then a solution $(\bar{x}, \bar{\lambda}, \bar{\gamma})$ to (D_2^*) or $(\bar{x}, \bar{\lambda})$ to (D_1^*) is such that $\bar{x}$ solves (P) and $z = z_2 = z_1$.

PROOF: Under the hypotheses of the theorem $L_2(x, \bar{\lambda}, \bar{\gamma})$ is strictly concave in x and thus the maximizer $\bar{x}$ of $L_2(x, \bar{\lambda}, \bar{\gamma})$ is unique, so by Theorem 9 of Geoffrion [5] $\bar{x}$ solves (P) and $z = z_2$. Since $(\bar{x}, \bar{\lambda})$ is feasible for (D_1^*), every (x, λ) feasible for (D_1^*) is feasible for (D_2^*), and the objective functions of (D_1^*) and (D_2^*) are the same, $(\bar{x}, \bar{\lambda})$ solves (D_1^*) and $z_2 = z_1$. A similar argument holds for $L_1(x, \bar{\lambda})$.

Q.E.D.

III. AN ALGORITHM

Two variations of a cutting plane algorithm, which approximate the dual programs (D_1^*) and (D_2^*), are offered. The basic algorithm was originally presented by Dantzig [1, ch. 24] as a column generation procedure, and the dual of that procedure has been considered by Zangwill [14, ch. 14] and Eaves and Zangwill [2]. The basic algorithm optimizes a sequence of linear master programs and either an unconstrained quadratic function or a quadratic function over the nonnegative orthant. The algorithm converges if either (a) V is negative definite, (b) at each iteration some $\lambda_i^k > 0$ and the corresponding W_i is positive definite, or (c) the set of all trial points x^k is compact. The optimality test is that the values of the objective functions of the primal and the dual are equal. The convergence proof is given in the appendix and is essentially that offered by Zangwill [14, ch. 14]. Eaves and Zangwill [2] have developed conditions under which certain of the constraints in steps 2_I and 2_{II} may be dropped.

ALGORITHM:

1. Choose a finite x^0 feasible* for (P) where the superscript indicates the iteration number.

Variation I:

2_I: At the kth iteration solve the linear program,

$$\begin{aligned} &\underset{\lambda, \mu}{\text{minimize}}\ \mu \\ &\text{subject to } L_1(x^{j_I}, \lambda) - \mu \leq 0, \qquad j_I = 0, \ldots, k_I - 1 \\ &\lambda \geq 0. \end{aligned}$$

Denote the solution by $(\mu^{k_I}, \lambda^{k_I})$.

3_I: Maximize $L_1(x, \lambda^{k_I})$

subject to $x \geq 0$.

*To prove convergence of the algorithm, the initial trial point x^1 should not be on the boundary of a constraint, since then an infinite λ_i may be optimal in step 2. In practice, any feasible x^1 may be used and a very large value for λ_i used if necessary. A strictly interior point for (P) may be generated by the techniques used with barrier methods (see [3]).

Denote the optimal solution† by x^{kI}. If $\mu^{kI} = L_1(x^{kI}, \lambda^{kI})$, x^{kI} solves (P). Otherwise go to step 2_{I}.

Variation II:

2_{II}: At the kth iteration solve the linear program,

$$\begin{aligned} &\underset{\lambda,\mu,\gamma}{\text{minimize}}\ \mu \\ &\text{subject to } L_2(x^{j_{II}}, \lambda, \gamma) - \mu \leqslant 0, \qquad j_{II} = 0, \ldots, k_{II} - 1 \\ &\qquad\qquad \lambda \geqslant 0, \qquad \gamma \geqslant 0. \end{aligned}$$

Denote the optimal solution by $(\mu^{kII}, \lambda^{kII}, \gamma^{kII})$.

3_{II}: Maximize $L_2(x, \lambda^{kII}, \gamma^{kII})$

subject to $x \epsilon E^n$.

Denote the solution by x^{kII}. If $\mu^{kII} = L_2(x^{kII}, \lambda^{kII}, \gamma^{kII})$, x^{kII} solves (P). Otherwise go to step 2_{II}.

The advantage of Variation II is that if V is negative definite or if at an iteration $\lambda_i^{kII} > 0$ and W_i is positive definite, x^{kII} is given by the closed form

$$x^{kII} = \tfrac{1}{2}[V - \Sigma \lambda_i^{kII} W_i]^{-1}[A'\lambda^{kII} - c - \gamma^{kII}], \tag{2}$$

which may be easier to obtain than x^{kI} in step 3_{I}. The linear program in 2_{II}, however, contains more variables than that in 2_{I}. Since it may be difficult to predict whether a $\lambda_i^{kII} > 0$ for some W_i positive definite, Variation II may be used whenever $[V - \Sigma \lambda_i^{kII} W_i]$ is nonsingular and then switching to Variation I if it is singular.

The programs in step 2_{I} and 2_{II} approximate the duals (D_1), (D_1^*), (D_2), and (D_2^*), in the sense that

$$\underset{\lambda \geqslant 0}{\text{minimize}} \left[\underset{j_I = 1,\ldots,k_I - 1}{\text{maximize}} L_1(x^{j_I}, \lambda)\right] \tag{3}$$

approximates D_1. Let $\mu = \text{maximum } L_1(x^{j_I}, \lambda)$, so $L_1(x^{j_I}, \lambda) \leqslant \mu$ for all j_I. The program (3) is thus equivalent to that in 2_{I}. Alternatively, observe that (D_2^*) is equivalent to

$$\begin{aligned} &\underset{x,\lambda,\mu,\gamma}{\text{minimize}}\ \mu \\ &\text{subject to } -x'Vx + \Sigma \lambda_i x' W_i x + \lambda' b - \mu \leqslant 0 \\ &\qquad c + 2Vx - A'\lambda - 2\Sigma \lambda_i W_i x + \gamma = 0 \\ &\qquad \lambda \geqslant 0, \qquad \gamma \geqslant 0. \end{aligned}$$

Postmultiply the second constraint (transposed) by x and substitute into the first constraint and the program in step 2_{II} is obtained.

The primal program (P) may be written as

$$\max_{x \geqslant 0} \min_{\lambda \geqslant 0} L_1(x, \lambda) \tag{P_1}$$

or as

† Since $L_1(x, \lambda^{kI})$ is concave in x and the only constraints are $x \geqslant 0$, the procedure of Theil and van de Panne [13] may be used.

$$(P_2) \qquad \max_{x \epsilon E^n} \min_{\substack{\gamma \geq 0 \\ \lambda \geq 0}} L_2(x, \lambda, \gamma).$$

Steps 3_{I} AND 3_{II} are thus approximations to (P_1) and (P_2), respectively, using λ^k or (λ^k, γ^k). The optimality test is that the objective function for the primal $L_1(x^k, \lambda^k)$ (or $L_2(x^k, \lambda^k, \gamma^k)$) equals the value of the objective function for the dual μ^{kI} (or μ^{kII}).

At any iteration the optimal value of the objective function of (P) is bounded above by $L_1(x^k, \lambda^k)$ (or $L_2(x^k, \lambda^k, \gamma^k)$) by the weak duality Theorem 2 [4]. To determine a lower bound on the optimal value of the objective function, let $t^j \geq 0$ be the dual variable associated with the jth constraint of the linear program in step 2_{I}. Clearly $\sum_{j=1}^{k-1} t^j(c'x^j + (x^j)'Vx^j) = \mu^k$ and $\sum_{j=1}^{k-1} t^j = 1$ by the duality theory for linear programs. Since the objective function is concave, by Jensen's inequality $\mu^k \leq c'(\Sigma t^j x^j) + (\Sigma t^j x^j)'V(\Sigma t^j x^j)$ where the latter is less than or equal to the optimal value of the objective function.

IV. AN EXAMPLE – PRODUCTION PLANNING

Schramm and Damon [10] have proposed a model of production planning involving quadratic constraints which stimulated the following example. Consider a firm that makes three decisions in a period: x_n the number of workers employed in period n, z_n the number of hours worked per employee, and y_n the units sold in period n. Output K_n is given by a quadratic production function

$$K_n = \alpha_n x_n z_n - \beta_n (x_{n-1} - x_n)^2 - \eta_n (z_{n-1} - z_n)^2.$$

The first term is a linear function of the total hours worked and the last two terms represent inefficiencies due to changes in the work force and the hours worked, respectively. The firm carries an inventory I_n between periods, $I_n \equiv I_{n-1} + K_n - y_n$, which is restricted to be nonnegative yielding a constraint quadratic in x_n and z_n. In addition, there may be a restriction on the change in hours worked per man $(z_{n-1} - z_n)^2 \leq B_n$.

The objective is to maximize profit over a finite horizon of N periods. Assume that revenue is given by $[P_n = (a_n - b_n y_n)] \cdot y_n$, where P_n is the price in period n. Costs are the variable cost of labor $\gamma_n x_n z_n$, where γ_n is the wage rate, the cost of hiring and firing $\delta_n (x_{n-1} - x_n)^2$, the cost of changing the hours worked $\xi_n (z_{n-1} - z_n)^2$, and the inventory holding cost $h_n I_n$. The firm's program then is

$$(4) \qquad \max \sum_{n=1}^{N} [(a_n - b_n y_n) \cdot y_n - \gamma_n x_n z_n - \delta_n (x_{n-1} - x_n)^2 - \xi_n (z_{n-1} - z_n)^2 - h_n I_n],$$

subject to

$$I_n \equiv I_{n-1} + \alpha_n x_n z_n - \beta_n (x_{n-1} - x_n)^2 - \eta_n (z_{n-1} - z_n)^2 - y_n \geq 0$$
$$(z_{n-1} - z_n)^2 \leq B_n$$
$$y_n, x_n, z_n \geq 0$$
$$n = 1, \ldots, N.$$

The initial conditions I_0, z_0, x_0 are assumed fixed.

As an example, (4) was solved using the algorithm for $N=1$ and the following parameters:

$$\beta=1; \quad \eta=1; \quad \alpha=.2; \quad a=20; \quad b=1; \quad \gamma=.2; \quad x_0=1$$

$$z_0=40; \quad \delta=1; \quad B=10; \quad c=45; \quad I_0=5; \quad h=5; \quad \xi=1.$$

The iterations using variation *I* are:

Iteration k	y^{k-1}	x^{k-1}	z^{k-1}	μ^k	λ_1^k	λ_2^k
1	20.0	1.0	40.0	−4.5	0	0
2	10.5	0.0	40.0	84.5172	2.6897	0
3	8.905	2.495	40.093	91.6621	1.5904	0
4	9.455	1.173	40.005	93.9250	1.2423	0
5	9.629	0.408	39.994	94.7028	1.4010	0
6	9.550	0.792	39.996	94.8446	1.3184	0
7	9.591	0.601	39.994	94.9073	1.2818	0
8	9.609	0.501	39.994	94.9083	1.3022	0
9	9.599	0.561	39.994	94.9092	1.3209	0
10	9.590	0.607	39.994			

The optimal solution is (approximately) $(y, x, z) = (9.590, 0.607, 39.994)$.

APPENDIX

Zangwill [14] has developed a general theory of algorithmic convergence which in many cases, including this one, simplifies proofs of convergence. Only the proof for Variation I will be given here. For cutting-plane algorithms, he has demonstrated that it is sufficient to show the following:

1. All test points (μ^k, λ^k) are contained in a compact set and all points x^k generated by the algorithm at an iteration are contained in a compact set.

2. Define $Y^k=\{\mu, \lambda \mid f(x^j)+\lambda' g(x^j)-\mu \leq 0;\ j=1, \ldots, k-1,\ \lambda \geq 0\}$, where $f(x^j)=c'x^j+x^{j\prime}Vx^j$ denotes the objective function in (P) and $g(x^j)=b-Ax^j-(x^{j\prime}W_1x^j, \ldots, x^{j\prime}W_mx^j)'$ denotes the constraints in (P) evaluated at the solution at the jth iteration. Then for the optimality test $\mu^k=L(x^k, \lambda^k)$, a point satisfying the test implies $(\mu^k, \lambda^k) \in Y^{k+1}$.

3. The map $\Delta(\lambda^k)$, which is the calculation of x^k in step 3 of the algorithm, is closed.* Also, $f(x)$ and $g(x)$ are continuous.

4. For (μ^k, λ^k) not satisfying the optimality test, and for x^k, it must be shown that

$$(\mu^k, \lambda^k) \notin H(x^k)=\{(u, \lambda) \mid f(x^k)+\lambda' g(x^k)-\mu \leq 0,\ \lambda \geq 0\}, \qquad \text{and } Y^k \cap H(x^k) \neq \varphi,$$

where φ is the empty set.

If the four conditions are satisfied, the sequence $\{x^k\}$ has a limit point that solves (P).

CONVERGENCE THEOREM: If either (a) V is negative definite, or (b) the set of points x^k is

---

*A map A from a set into its power set is closed at a limit point x^* of a sequence x^k, $k \in K$ (an infinite subsequence of integers), if (a) $x^k \rightarrow x^*$, $k \in K$, (b) $y^k \rightarrow y^*$, $k \in K$, where y^k is the next point generated by the algorithm $A(x^{k-1})$ when x^{k-1} is not optimal, and (c) y^k is a possible value of x^k for a computed x^{k-1}, $k \in K$, together imply $y^* \in A(X^*)$. In this case A is the algorithm generating successive values of x^k in step 3.

compact, or (c) at every iteration some $\lambda_i^k > 0$ and the corresponding W_i is positive definite, and if there exists an x^0 such that $g(x^0) = b - Ax^0 - (x^{0\prime}W_1x^0, \ldots, x^{0\prime}W_mx^0) > 0$, the sequence $\{x^k\}$ converges to a limit point that solves (P).

PROOF: The proof involves the verification of points 1–4.

1. To show that the (μ^k, λ^k) are contained in a compact set, it is sufficient to demonstrate that they are bounded. As indicated in section III, μ^k is bounded above by the optimal value of the objective function of (P). By assumption there exists a point x^0 such that $g(x^0) > 0$. Then for $\lambda = 0, f(x^0) \leq \mu^k$ from the program in step 2_1. Therefore, μ^k is bounded. To show that λ_i^k, $i = 1, \ldots, m$, are bounded, observe that $\lambda_i g_i(x^0) \leq \lambda' g(x^0)$, since $\lambda \geq 0$ and $g(x^0) > 0$. From the program in step 2_1 $\lambda' g(x^0) \leq \mu - f(x^0) \leq f(\hat{x}) - f(x^0)$, where $\hat{x}$ is optimal for (P). Then

$$0 \leq \lambda_i \leq \frac{f(\hat{x}) - f(x^0)}{g_i(x^0)},$$

and λ_i is bounded for all i. It will be shown in 3 below that the solutions x^k generated in step 3 will be in a compact set in E^n.

2. At the optimal solution $x^k = \hat{x}$, $\mu^k = f(x^k) + \lambda^{k\prime} g(x^k)$, which is clearly in Y^{k+1}.

3. The map $\Delta(\lambda^k)$ is the calculation of x^k in step 3_1. To show that it is closed, it is sufficient (Zangwill, [14, ch. 7]) to show that the set of feasible solutions to the program in step 3_1 is compact or equivalently in this case that the set is bounded. Under assumptions (a) or (c) above, the function $L_1(x, \lambda^k)$ is strictly concave, and a strictly concave, quadratic function must take on its maximum at a finite point. A bound on the set of solutions x^k thus exists and the map $\Delta(\lambda^k)$ is closed. If neither (a) nor (c) holds, (b) is required. Clearly, $f(x)$ and $g(x)$ are continuous.

4. If (μ^k, λ^k) does not satisfy the optimality test, then $\mu^k < f(x^k) + \lambda^{k\prime} g(x^k)$ so $(\mu^k, \lambda^k) \notin H(x^k)$. Finally, it is necessary to show there exists a point $(\mu, \lambda) \in Y^k \cap H(x^k)$. Clearly, $(\lambda = 0, f(\hat{x}) = \mu)$ is such a point. The algorithm converges.

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INVENTORY ALLOCATION AMONG AN ASSEMBLY AND ITS REPAIRABLE SUBASSEMBLIES*

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ABSTRACT

In this paper we consider a major assembly composed of two or more subassemblies. The failure of any subassembly causes the major assembly to not function. Every failed subassembly is repaired or replaced. A total investment in stocks of spare components is to be distributed among the various subassemblies and the major assembly so as to provide the best possible customer service. This is a complicated problem: relevant factors are the failure rates, unit costs, and repair times of the various components. For the case of Poisson failures, a heuristic solution is developed which is a compromise between theoretical optimality and practical usefulness.

I. INTRODUCTION

We consider the situation where we have a major assembly (or line-repairable unit) having several component subassemblies (or modules). From time to time, major assemblies break down because of the failures of subassemblies. All failed units are repaired, but the detection and repair times are random variables. Cannibalization is used. Given a specified budget, the problem is to allocate this budget among spares of the major assembly and its subassemblies in such a way as to make customer service as high as possible. The budget allocation decision depends upon the unit costs of the various subassemblies and the entire assembly, the failure rates of the various components, and statistics of the detection and repair times.

For the case of no cannibalization, an optimal allocation procedure has been developed by Moore et al. [7]. Demmy [4] has extended their analysis to the case where the detection and repair times are also decision variables whose values can be influenced by the amount of resources invested in detection and repair facilities. For the cannibalization situation (considered here), Sherbrooke [9] has developed a *descriptive* model which gives the probability distribution of the number of inoperable assemblies for a given allocation of a budget to the various types of spares. Sherbrooke's results are not amenable to the development of a *normative* model because of the nonseparability of the mathematical expression for the assembly availability.

In the current paper we start with Sherbrooke's descriptive results. For a particular measure of customer service, namely the ready rate,† we show that the objective function (the ready rate) is separable for the special case of zero spares of the assembly itself. This allows us to obtain the best alloca-

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† The ready rate is the fraction of time that there are zero customer backorders.

tion of a given budget among spares of the subassemblies conditional upon no spares of the entire assembly. Then we use a heuristic argument to generate, from this special solution, a set of solutions where spares of the assembly are allowed. This set, although quite reasonable in size (compared with the number of feasible solutions), has contained the optimal solution (or one very close to it) for all numerical examples on which the heuristic algorithm has been tested.

II. THE PROBLEM SETTING AND THE BASIC MODEL

As we shall be starting with the descriptive results developed by Sherbrooke, it is appropriate to describe the problem setting for which he developed his model. In this connection Figure 1 should be of assistance to the reader.

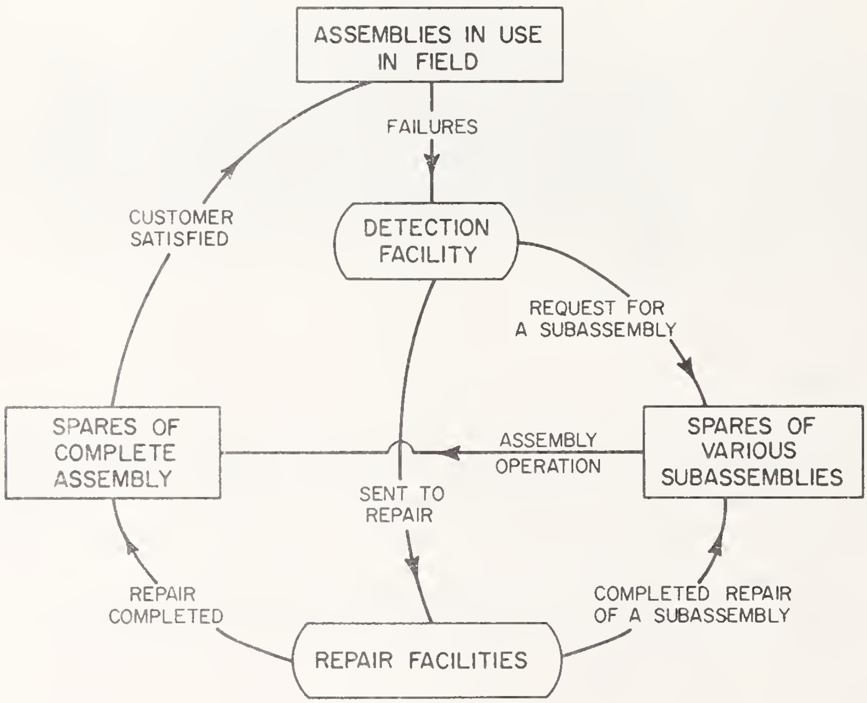

FIGURE 1. Schematic of the Problem Setting

There are a large number of major assemblies in use in the field. With an on-going system there are inventories of spare assemblies and subassemblies, some of which may be at a zero level.

The breakdown of an assembly causes it to arrive at the stocking location. If a complete assembly is available, it is exchanged for the failed one and the customer waits a negligible length of time. If a complete assembly is not available, the customer must wait. The failed assembly goes to a checkout or detection station where the failed subassembly is detected (the detection time being a random variable). Once the failed subassembly is identified, it is removed and replaced if a spare is available (this spare can be cannibalized from another assembly waiting for some other component to be repaired). If such a replacement is possible, the customer (or possibly an earlier one that generated the other assembly involved in the cannibalization) is satisfied. Each failed unit is sent off for repair. The model allows a probabilistic assignment of each repair to one of two or more repair locations. The repair time is a random variable which can depend upon both the unit involved and the location at which

the repair takes place. Also, the model can handle the fact that some failed units are replaced by new units rather than being repaired. Because of the possibility that the local detection facility cannot identify the failed component, the model allows for a certain percent of the assemblies having to be sent off for repair as whole units.

The completion of repair of a subassembly may trigger the preparation of a complete assembly. Furthermore, the arrival of a complete assembly into its inventory location leads to the satisfying of an outstanding customer backorder, if any exists.

Incidentally, the model allows for an assembly requiring more than one unit of a particular subassembly, e.g., a truck requiring four tires.

III. ASSUMPTIONS

As Sherbrooke [9] has elaborated on the assumptions, we shall not repeat his arguments here. Needless to say, the validity of each assumption was checked with the personnel of the Defence Research Board (DRB) of Canada for whom the study was conducted.

1. There are a large number of assemblies in use in the field (infinite source population).
2. Failures of subassemblies occur according to independent Poisson processes.
3. A single failure is sufficient to make the entire assembly inoperable.
4. The checkout time for an assembly is independent of the number of assemblies already in the checkout process (infinite channel queueing assumption).
5. A similar assumption is made for the subassembly repair time or assembly repair time.
6. The means of the checkout time distribution and repair (replacement) time distributions must be finite but no assumption is made about the forms of the distributions.
7. Complete cannibalization exists.

IV. SHERBROOKE'S DESCRIPTIVE RESULTS

Notation

(Some further notation will be introduced later in the paper as it is needed)

a_i — number of units of subassembly i required per assembly $(i=1,2, \ldots, n)$
$\bar{B}$ — expected number of assemblies backordered at a random point in time
n — number of subassembly types
p_0 — prob {whole assembly has to be sent off for repair}
p_i — prob {subassembly i needs repair | assembly has failed} $(i=1,2, \ldots, n)$
$P(x)$ — prob {x assemblies are unavailable because of being checked out or being under repair as whole units}
$Q(x)$ — prob {x assemblies are unavailable because of lack of subassemblies}
$R(x)$ — prob {x *or less* assemblies are unavailable because of lack of subassemblies}
s_0 — stock level of major assembly
s_i — stock level of subassembly i $(i=1,2, \ldots, n)$
t_0 — *average* repair time for an assembly when it has to be sent out as a unit
t_i — average repair time of subassembly i $(i=1,2, \ldots, n)$
T — average checkout time
λ — failure rate of the entire assembly

λ_o — parameter used in computing $P(x)$
λ_i — parameter used in computing $R(x)$ $(i=1,2, \ldots, n)$
$\phi(b)$ — prob {*b or less* assemblies are backordered}
$\chi(b)$ — prob {exactly b assemblies are backordered}.

RESULTS:

(1) $$\phi(b)=\sum_{y=0}^{s_0+b}\sum_{x=0}^{y} P(x)\, Q(y-x),$$

(2) where $$P(x)=\frac{\exp(-\lambda_o)\lambda_o^x}{x!},$$

(3a) $$Q(x)=R(x)-R(x-1) \qquad (x=1,2, \ldots),$$

(3b) $$Q(0)=R(0), \text{ and}$$

(4) $$R(x)=\prod_{i=1}^{n}\left(\sum_{k=0}^{s_i+a_i x}\frac{\exp(-\lambda_i)\lambda_i^k}{k!}\right),$$

with

(5) $$\lambda_o=\lambda T+\lambda p_o t_o$$

and

(6) $$\lambda_i=\lambda p_i t_i \qquad (i=1,2, \ldots, n).$$

Also

(7) $$\bar{B}=\sum_{b=1}^{\infty} b\chi(b),$$

where

(8) $$\chi(b)=\phi(b)-\phi(b-1) \qquad (b=1,2, \ldots).$$

The key element in Sherbrooke's proof of the above results is Palm's [8] Theorem which states that for an infinite channel queueing system, if the demand is Poisson with rate μ and the service time distribution has a mean Y independent of the number of customers being served, then the steady-state probabilities for the number of customers in the system are Poisson with mean μY. (This result is independent of the form of the service time distribution.)

Because of the complex nature of $R(x)$ the objective function $\overline{B}$ cannot be written as a separable function of the control variables $s_o, s_1, s_2, \ldots, s_n$. In nonmathematical terms, because of the cannibalization property, the benefit derived from s_i spares of subassembly type i depends very much upon the values at which the other s's are set. This nonseparable property is what prevented Sherbrooke from developing a normative model for prescribing the values of the s's.

V. MATHEMATICAL PROBLEM STATEMENT

Within the problem context posed by the DRB personnel it is reasonable to work with "ready rate" as the service criterion as opposed to "expected backorders." An excellent description of different measures of service is provided by Brooks et al. [3]. As mentioned earlier, the ready rate is defined to be the probability of zero backorders at a random point in time. If we denote the ready rate by α, we have

$$\alpha=\phi(0).$$

This, together with Equation (1), allows us to write

$$\alpha=\sum_{y=0}^{s_0}\sum_{x=0}^{y} P(x)Q(y-x).$$

By use of Equation (3), this simplifies to

$$\alpha=\sum_{x=0}^{s_0} P(s_o-x)R(x). \tag{9}$$

If we let v_o be the unit value of a complete assembly, v_i be the unit value of subassembly i $(i=1, 2, \ldots, n)$, and D be the budget to be allocated among spares, then the mathematical statement of our problem is:

Select $s_o, s_1, \ldots, s_n$ so as to

$$\text{Maximize } \alpha=\sum_{x=0}^{s_0} P(s_o-x)R(x) \tag{10}$$

$$\text{subject to } \sum_{i=0}^{n} s_i v_i \leq D \tag{11}$$

$$\text{and } s_i=0, 1, 2, \ldots \qquad (i=0, 1, 2, \ldots, n). \tag{12}$$

Equation (11) is the budget constraint while Equation (12) follows from the nonnegative integer nature of stocks.

Unfortunately the objective function of Equation (9) is still in a nonseparable form. What we have is a nonlinear integer programming problem. The state of the art precludes exact solution of this problem on a repetitive basis (there are many types of assemblies in the inventory under consideration). Fortunately, a special case can be solved quite easily as we shall see in the next section.

VI. MAXIMIZING THE READY RATE FOR THE SPECIAL CASE OF NO SPARES OF THE TOTAL ASSEMBLY

Consider the situation where we are constrained to having no spares of the entire assembly, i.e., $s_0=0$. In this case Equations (2), (4), and (9) give us the ready rate as

$$\alpha = \exp\left(-\sum_{j=0}^{n} \lambda_j\right) \prod_{i=1}^{n} \left(\sum_{k=0}^{s_i} \lambda_i^k/k!\right). \tag{13}$$

Now, the problem reduces to

$$\text{Maximize } \alpha \text{ (of Equation (13))}$$

subject to the constraints

$$\sum_{i=1}^{n} s_i v_i \leq D$$

and

$$s_i = 0, 1, 2, \ldots \qquad (i=1, 2, 3, \ldots, n).$$

It is clear from Equation (13) that the objective function is now a separable function of the decision variables, the s_i's. We shall discuss three possible ways of solving this special ($s_0=0$) optimization problem.

Method 1 – Dynamic Programming

We have a separable objective function with integer variables and a single resource constraint. An obvious solution procedure is dynamic programming. We shall not go into the details because such a problem has been adequately treated in the literature (see, e.g., Bellman and Dreyfus [2]).

As long as the budget constraint D is not too large, the dynamic programming approach is quite efficient. However, when one or more of the v_i's are small and D is quite large, the computational requirement can become prohibitive. In such a situation, as we shall see, Method 3 is preferable.

Method 2 – Marginal Allocation

Maximizing the logarithm of α will also maximize α. Therefore, our problem can be equivalently written as

$$\text{Maximize } \ln \alpha = C' + \sum_{i=1}^{n} \ln\left(\sum_{k=0}^{s_i} \lambda_i^k/k!\right), \tag{14}$$

$$\text{subject to } \sum_{i=1}^{n} s_i v_i \leq D \tag{15}$$

$$\text{and } s_i = 0, 1, 2, \ldots \qquad (i=1, 2, \ldots, n). \tag{16}$$

Suppose we let

$$h_i(s_i) = \ln\left(\sum_{k=0}^{s_i} \lambda_i^k/k!\right). \tag{17}$$

Fox [6] has shown that, if $h_i(s_i)$ is strictly concave for each i ($i=1, 2, \ldots, n$), then marginal allocation will generate undominated solutions. An undominated solution is one such that any other solution producing a better value of the objective function must use more of the resource than does the undominated solution. Marginal allocation means that at any stage in the solution we allocate the next unit to the item i which maximizes

$$\frac{h_i(s_i+1) - h_i(s_i)}{v_i},$$

where s_i is the amount already allocated to item i. The proof of the strict concavity of $h_i(s_i)$ is shown in appendix A.

The generation of only undominated solutions does not guarantee that one obtains the best solution for a specified budget. To illustrate, suppose the budget D was 100. The marginal allocation procedure might generate two consecutive undominated solutions that straddled 100 (e.g., 97 and 104). Neither of these would give us the optimal solution for a budget of 100. As we shall see in section VII, this is not a serious problem because our heuristic algorithm does not require that the $s_o=0$ problem be solved for a budget of exactly D.

The marginal allocation procedure is simpler than dynamic programming. In fact, with suitable tables of partial sums of the Poisson distribution (see, e.g., Abramowitz and Stegun [1]) it can be done manually. However, again with a large value of D and one or more small values of v_i the procedure can be lengthy. The third approach, to now be discussed, does not have this undesirable characteristic.

Method 3 – Lagrange Multiplier Approach

Everett [5] has advocated a generalized Lagrange multiplier approach which is directly applicable to our problem. To solve a problem of the form depicted by Equations (14), (15), and (16), he has shown that we can proceed as follows:

Form the Lagrangian,

$$H = C' + \sum_{i=1}^{n} \ln\left(\sum_{k=0}^{s_i} \lambda_i^k/k!\right) - \gamma \sum_{i=1}^{n} s_i v_i. \tag{18}$$

Then, for any given value of the Lagrange multiplier, γ, we find the set of s_i's that maximize H. Such a solution is undominated (as defined in the preceding subsection). The set of s_i's selected implies a value of the budget used, namely

$$\sum_{i=1}^{n} s_i v_i.$$

γ is adjusted until the budget used comes sufficiently close to D (the larger γ is, the smaller will be the budget used).

It is clear from Equation (18) that H is a separable function of the s_i's. Furthermore appendix A shows that the portion which depends upon a specific s_i is a concave function of that s_i.

Suppose we let

$$\Delta H(s_i) = H(s_i+1) - H(s_i). \tag{19}$$

From Equation (18) we have

$$\Delta H(s_i) = \ln\left[\left(\sum_{k=0}^{s_i+1} \lambda_i^k/k!\right)\Big/\left(\sum_{k=0}^{s_i} \lambda_i^k/k!\right)\right] - \gamma v_i. \tag{20}$$

For a concave function of a discrete random variable the maximum is achieved at the s_i satisfying the following two inequalities,

$$\Delta H(s_i-1) \geqslant 0$$

and

$$\Delta H(s_i) \leqslant 0.$$

Use of Equation (20) gives the two conditions

$$\ln\left[\left(\sum_{k=0}^{s_i} \lambda_i^k/k!\right)\Big/\left(\sum_{k=0}^{s_i-1} \lambda_i^k/k!\right)\right] \geqslant \gamma v_i, \tag{21}$$

and

$$\ln\left[\left(\sum_{k=0}^{s_i+1} \lambda_i^k/k!\right)\Big/\left(\sum_{k=0}^{s_i} \lambda_i^k/k!\right)\right] \leqslant \gamma v_i. \tag{22}$$

With a simple computer program, given values of γ, v_i, and λ_i, there is no problem in finding the s_i that satisfies the inequalities (21) and (22). Alternatively, a simple graphical lookup can be used once curves of the type shown in Figure 2 have been developed once-and-for-all from Equations (21) and (22). To illustrate, suppose $\gamma = 0.03$, $v_i = 2$ (therefore $\gamma v_i = 0.06$) and $\lambda_i = 0.8$. Figure 2 indicates that the maximizing value of s_i is 2.

VII. MAXIMIZING THE READY RATE SUBJECT TO A BUDGET CONSTRAINT

In the previous section we have indicated that any one of three methods can be used to maximize the ready rate under the condition of no spares of the total assembly, i.e., $s_o = 0$. In general, however, the overall solution that maximizes the ready rate subject to a budget constraint will contain one or more spares of the total assembly. As indicated in section V, we cannot directly solve the overall maximization problem because of the nonseparability of the objective function. Therefore, we resort to the use of a heuristic which derives from the optimum solution for $s_o = 0$, a set of solutions (with $s_o > 0$) to be tested. The best of these solutions is the one to be selected.

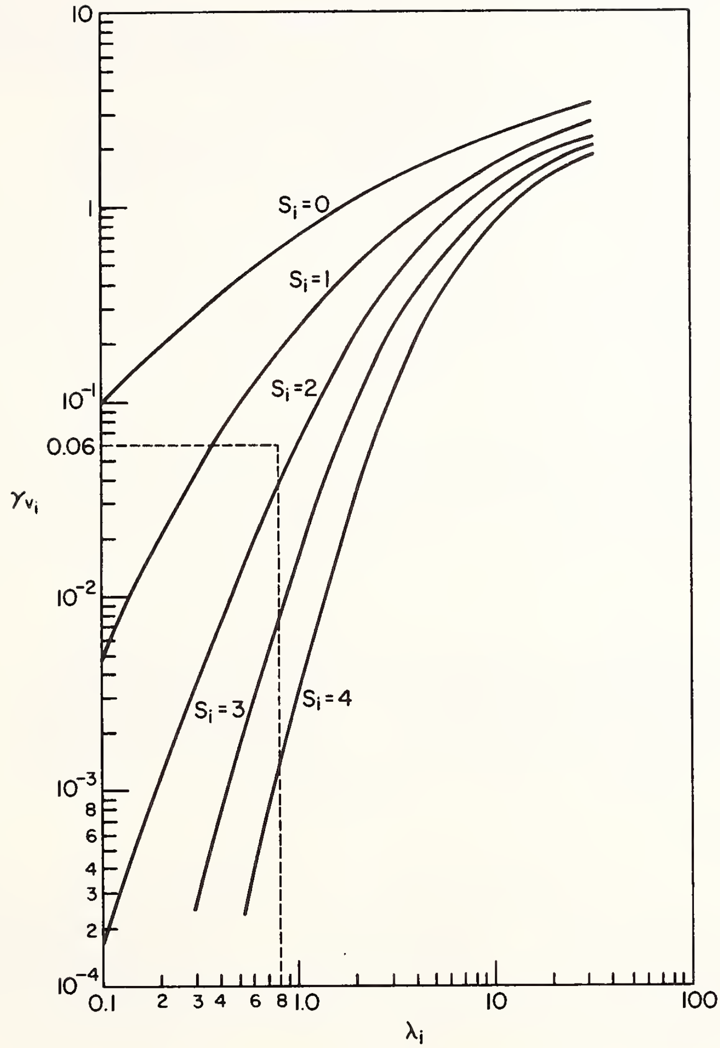

FIGURE 2. Graphical Aid for Lagrange Multiplier Approach

Numerical Example

To illustrate the ideas in this section we shall use the numerical example shown in Table I.

TABLE I. *Item Data for Numerical Example*

Item i	v_i	λ_i	a_i	
0	10	3.0	—	
1	2	1.0	1	Budget, $D=29$
2	2	1.5	1	
3	1	0.5	1	

Using the dynamic programming approach, the best solution for $s_o=0$ is given in Table II.

TABLE II. *Best Solution for* $s_0=0$

Item i	v_i	Allocation (s_i)	Remaining budget after allocation
0	10	0 (imposed)	29
1	2	5	19
2	2	7	5
3	1	5	0
		($\alpha=0.0497$)	

The Heuristic for Finding the Best Solution for a Specific Nonzero s_o Value

Suppose we look at a situation where a nonzero value of s_o is used. Assume that we have sequentially allocated amounts to items $1,2,\ldots,i-2$ and $i-1$. The basic component of the heuristic can be stated as follows:

$$\frac{\text{(Amount to now allocate to item } i)}{\text{(Amount allocated to item } i \text{ under } s_o=0 \text{ constraint)}} \simeq \frac{\text{(Budget remaining under current allocation)}}{\text{(Budget remaining at same stage under } s_o=0 \text{ allocation)}}$$

Cross-multiplication gives

$$\underbrace{\text{(Amount to now allocate to item } i)}_{\text{denote by } s_i'} \simeq \text{(Amount allocated to item } i \text{ under } s_0=0) \cdot \frac{\text{(Budget remaining under current allocation)}}{\text{(Budget remaining at same stage under } s_o=0 \text{ allocation)}} \tag{23}$$

We must have an integer number of units. Therefore, the algorithm considers the two integer values surrounding the result of Equation (23). If there are n subassemblies, we thus generate 2^{n-1} solutions* for each value of s_o tested. For each such solution the ready-rate, α, is evaluated, using Equation (10) together with Equations (2) and (4). The best of these has an α value denoted by $\alpha\max(s_o)$.

Numerical Example

To illustrate, let us consider the numerical example using $s_o=2$. The remaining budget is thus $(29-2\times 10)$ or 9. Using Equation (23) for item 1, we have

$$s_1' \simeq 5 \cdot \frac{9}{29} = 1.55.$$

Therefore, the two possible integers are 1 and 2.

*Each of the two solutions for item 1 has two solutions for item 2, etc. except for the last (nth) subassembly where, obviously, the best strategy is to use up as much as possible of the remaining budget.

Turning to item 2 we now have two possibilities to consider, one for each value of s_1. Consider the $s_1=1$ case. Here Equation (23) gives

$$s_2'=7\cdot\frac{(9-1\times 2)}{19}=2.6.$$

Therefore, the two possible integers are 2 and 3. Similarly, for $s_1=2$, we find that $s_2=1$ or 2.

Moving on to item 3 we must look at each of the four possible (s_1, s_2) situations. Because item 3 is the last subassembly, we allocate as much as we can of the remaining budget to it. The four possible solutions are shown graphically in Figure 3. It is also seen that $\alpha\max(2)=0.4998$ with $s_o=2$, $s_1=2$, $s_2=2$ and $s_3=1$.

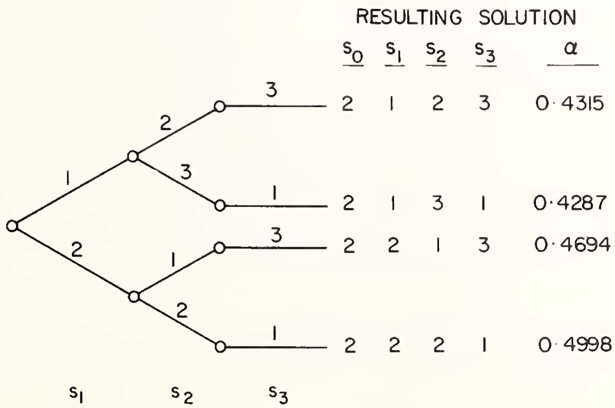

FIGURE 3. Solution Set of Numerical Example for $s_o=2$

Consideration of Different s_o Values

Conceptually one could find the $\alpha\max(s_o)$ value for each possible value of s_o. Graphically, a plot of the type shown in Figure 4 would result. The largest value of $\alpha\max(s_o)$ would indicate that the associated s_o and $s_i(i=1, 2, \ldots, n)$ values would be the overall solution to use. The majority of the numerical examples tested to date indicate the optimal solution occurs for a high value of s_o. Therefore, computationally it makes sense to start with s_o at its largest value and work downwards until $\alpha\max(s_o)$ first decreases. (In Figure 4 three values of s_o would have to be tested.)

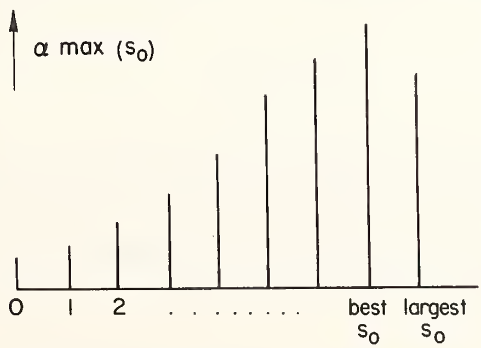

FIGURE 4. The Best α Value as a Function of s_o

Numerical Example

In the numerical example the largest possible value of s_o is the integer portion of $(29 \div 10)$ or 2. We have already found $\alpha\max(2)$ as 0.4998. Using the heuristic for the case of $s_o=1$ we find that $\alpha\max(1)$ is 0.2944. Therefore, the best solution for $s_o=2$ is preferable. As shown earlier this solution is

$$s_o=2,\ s_1=2,\ s_2=2,\ \text{and}\ s_3=1.$$

It turns out that this is, indeed, the overall optimal result for this example.

Too Large a Budget for the Case of $s_o=0$

The heuristic hinges on an approximate relationship between the best solution for $s_o=0$ and the best solution for any other s_o value. If the budget D is very large compared with the unit values of all items except the major assembly itself, then the optimal solution for $s_o=0$ can be essentially independent of unit values* and, hence, not likely comparable (in terms of relative proportions of various subassemblies) with the budget constrained situation when $s_o>0$. To avoid this problem we must use a small enough budget for the $s_o=0$ case. It should be emphasized that the heuristic algorithm does not require that exactly D be used for the $s_o=0$ allocation. (This is a point to which we alluded earlier when discussing that two of the $s_o=0$ methods did not necessarily give an allocation for a budget of exactly D.)

An easy way to limit the $s_o=0$ budget appropriately is to specify a smallest allowed incremental benefit of adding one more unit to an item, i.e., we specify a lower limit, ϵ, on

(24) $$h_i(s_i+1)-h_i(s_i)=\ln\left[\left(\sum_{k=0}^{s_i+1}\lambda_i^k/k!\right)\Big/\left(\sum_{k=0}^{s_i}\lambda_i^k/k!\right)\right].$$

In the marginal allocation approach (method 2) we terminate when an allocation would be made to an item such that the incremental benefit, as measured from Equation (24), was less than ϵ. In the Lagrange multiplier approach (method 3) it is seen from Equations (22) and (24) that we want not to exceed a budget where

$$\epsilon=\gamma v_i \qquad \text{for any } i.$$

Decreasing γ increases the budget. Therefore, we must not make γ smaller than

(25) $$\gamma=\max_i \frac{\epsilon}{v_i}.$$

Our procedure is thus to start with the γ given by Equation (25) and solve for the associated s_i's using Equations (21) and (22). If the resulting budget is less than D, we use it for the $s_o=0$ case. However, if the resulting budget is larger than D, we should increase γ until we get somewhat closer to D (in this case the negligible marginal benefit situation does not apply).

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*What happens here is that one or more items are used until there is negligible benefit from increasing their s_i's, quite independent of their unit costs.

A Special Property of the Optimal Solution

For the case of every a_i equal to unity and

$$v_o \leqslant \sum_{i=1}^{n} v_i$$

(i.e., the cost of the total assembly is not greater than the sum of the costs of the subassemblies) it is shown in appendix B that the optimal solution will have at least one $s_i(i>0)$ equal to zero. This does *not* necessarily mean that in the optimal solution s_o is as large as possible. The optimal solution might be to have s_o fairly low, one or more s_i's at zero levels and a few s_i's quite large.

Computationally this property could be useful. Suppose, that the $s_o=0$ solution gave one s_i lower than the rest. Then, it would be reasonable to assume that this s_i would be zero in the optimal solution. This would reduce by a factor of two the number of solutions to be tested by the algorithm.

The Case of $a_i > 1$

The set of solutions to be tested by the algorithm is seen to be independent of the a_i values. Certainly, at least in extreme cases, the optimal allocation should depend upon the a_i values. However, for each solution tested the α value is obtained using the actual a_i quantities. Interestingly enough, on all numerical tests performed, the optimal solution has been quite insensitive to the a_i values.

VIII. NUMERICAL RESULTS

The algorithm has been tested on a number of examples. The smaller ones are hypothetical but the larger ones were developed from military supply data. The 17 examples, together with the optimal solutions, are shown in the first eight columns of Table III. To get the true optimal solution we evaluated the objective function (the α of Equation (10)) for all solutions that could possibly be the optimum. We call such solutions boundary solutions, boundary meaning that it is impossible to get any closer to the budget constraint (i.e., no extra unit of any item can be added without exceeding the budget). Obviously, a nonboundary solution could not possibly be optimal because adding an extra unit of an item has to improve the objective function. The number of boundary solutions becomes prohibitively large for examples involving many items and a large budget; e.g., in Example 9a the number of boundary solutions with $s_o=3$ is 42,796 even though the budget is only 95. (For this reason most examples analyzed had rather small budgets.) In contrast, for the same example the algorithm tests only some 64 solutions with $s_o=3$.

TABLE 3. *Numerical Results*

Example number (1)	Budget (2)	Number of items (3)	Item i (4)	v_i (5)	λ_i (6)	a_i (7)	Optimum s_i (8)	Percent penalty in ready rate: Using full algorithm (9)	Percent penalty in ready rate: Using rounding in algorithm (10)
1a	30	4						0	0
			0	8	6	–	3		
			1	3	1	1	1		
			2	2	2	2	1		
			3	1	3	1	1		
1b	30	4						0	0
			0	same		–	3		
			1	as		1	1		
			2	ex. 1a		5	1		
			3			1	1		
1c	30	4						0	0
			0	same		–	3		
			1	as		1	1		
			2	ex. 1a		10	1		
			3			1	1		
1d	30	4						0	16.1
			0	same		–	3		
			1	as		3	0		
			2	ex. 1a		1	2		
			3			1	2		
1e	30	4						0	2.6
			0	same		–	3		
			1	as		1	0		
			2	ex. 1a		1	2		
			3			3	2		
2a	19	3						0	0
			0	5	1	–	1		
			1	3	10	1	4		
			2	2	1	1	1		
2b	19	3						4.1	4.1
			0	same		–	3		
			1	as		10	0		
			2	ex. 2a		1	2		
2c	19	3						5.9	5.9
			0	same		–	3		
			1	as		50	0		
			2	ex. 2a		1	2		
2d	19	3						0	0
			0	same		–	1		
			1	as		1	4		
			2	ex. 2a		50	1		

TABLE 3. *Numerical Results* — Continued

Example number	Budget	Number of items	Item i	v_i	λ_i	a_i	Optimum s_i	Percent penalty in ready rate	
								Using full algorithm	Using rounding in algorithm
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
3	11	5						0	0
			0	5	3.0	–	2		
			1	2	1.35	1	0		
			2	1	0.1	1	0		
			3	1	0.4	1	0		
			4	1	0.6	1	1		
4	13	5						0	0
			0				2		
			1	same			1		
			2	as			0		
			3	ex. 3			0		
			4				1		
5	11	5						0	0
			0	10	3.0	1	1		
			1	2	1.65	1	0		
			2	1	0.1	1	0		
			3	1	0.4	1	0		
			4	1	0.6	1	1		
6	11	5						0	0
			0	10	0	–	0		
			1	2	1.65	1	3		
			2	1	0.1	1	1		
			3	1	0.4	1	2		
			4	1	0.6	1	2		
7	29	4						0	0
			0	10	3.0	–	2		
			1	2	1.0	1	2		
			2	2	1.5	1	2		
			3	1	0.5	1	1		
8	460	10						0	0.4
			0	66	3.114	–	5		
			1	12	0.161	1	1		
			2	10	0.912	1	3		
			3	7	0.365	1	2		
			4	7	0.591	1	3		
			5	6	0.073	1	1		
			6	6	0.353	1	2		
			7	6	0.376	1	2		
			8	5	0.365	1	2		
			9	3	0.365	1	4		
9a	95	8						0	1.1
			0	20	0.625	–	3		
			1	6	1.688	1	3		
			2	5	0.625	1	1		
			3	2	0.125	1	1		

TABLE 3. *Numerical Results*—Continued

Example number (1)	Budget (2)	Number of items (3)	Item i (4)	v_i (5)	λ_i (6)	a_i (7)	Optimum s_i (8)	Percent penalty in ready rate: Using full algorithm (9)	Percent penalty in ready rate: Using rounding in algorithm (10)
			4	2	0.187	1	1		
			5	2	0.313	1	2		
			6	1	0.313	1	2		
			7	1	0.500	1	2		
9b	95	8						0	1.2
			0			–	3		
			1			10	2		
			2	some		1	2		
			3	as		1	1		
			4	ex. 9a		1	1		
			5			1	2		
			6			1	2		
			7			1	3		
							Grand averages	0.6	1.8

The results of using the full-blown heuristic algorithm are indicated in column (9) of Table III. The striking feature is that in 15 out of 17 examples the algorithm obtains the optimal solution. In the other two cases the errors (in terms of reduced ready rate because of not being at the optimal solution) are only 4.1 and 5.9 percent. Both of these cases involve a very large value of a_i, which is probably atypical of most physical situations.

As discussed earlier, for each value of s_o, the algorithm must test 2^{n-1} solutions where n is the number of subassembly types. For large n the computational requirements can become prohibitive. In such cases it might make sense to use a cruder version of the heuristic involving far less computations, namely rounding the noninteger s_i' value of Equation (23) to the nearest integer rather than allowing for both surrounding integers. If this is done, only a single solution need be tested for each s_o value. This very crude approach was tested on the 17 examples and the results are shown in column (10) of Table III. Interestingly enough, the optimal solution is still found in 10 out of the 17 cases. The average penalty (compared with the optimal solution) for using the crude algorithm is only 1.8 percent.

IX. SUMMARY

In this paper we have considered a complex problem in inventory control, namely the allocation of a budget to spares of an assembly and its repairable (replaceable) subassemblies. The approach has been to take advantage of being able to solve a constrained version ($s_o=0$) of the problem and to use this solution together with a heuristic rule to obtain a reasonably small set of solutions for the original (unconstrained) situation. The numerical examples indicate that the heuristic is robust enough to include the optimal solution (or one very close to it in performance) in the limited number of solutions in the aforementioned set. It is felt that this type of approach might be applicable to a number of other problems involving integer constraints.

One does not have any difficulty in suggesting possible extensions of the algorithm presented here. Obviously, a service measure other than the ready rate (e.g., average backorders) might be more appropriate for other applications. A second extension would be to the situation where there is more than one type of major assembly to which a total budget is to be allocated. An equivalent extension would be to the case of the same major assembly being stocked at more than one location. Sherbrooke [9] advocates the use of the METRIC model [10], developed for other purposes, to handle this type of extension.

ACKNOWLEDGEMENTS

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APPENDIX A—PROOF OF CONCAVITY OF $h_i(s_i)$ AND H

$h_i(s_i)$

For a function of a discrete variable, a sufficient condition for strict concavity is that the second difference is negative for all possible values of the variable. For our function*

$$h(s)=\ln\left(\sum_{k=0}^{s}\lambda^k/k!\right)$$

$$\Delta h(s)=h(s+1)-h(s)=\ln\left[\left(\sum_{k=0}^{s+1}\lambda^k/k!\right)\bigg/\left(\sum_{k=0}^{s}\lambda^k/k!\right)\right]$$

$$\Delta^2 h(s)=\Delta h(s+1)-\Delta h(s)$$

$$=\ln\left[\frac{\left(\sum_{k=0}^{s+2}\lambda^k/k!\right)\left(\sum_{k=0}^{s}\lambda^k/k!\right)}{\left(\sum_{k=0}^{s+1}\lambda^k/k!\right)^2}\right].$$

To prove that this is negative we need only show that the argument of the logarithm is less than unity.

Now, the argument of the logarithm can be written as

$$\frac{\left[1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\ldots+\frac{\lambda^s}{s!}\right]^2+\frac{\lambda^{s+1}}{(s+1)!}\left[1+\frac{\lambda}{s+2}\right]\left[1+\frac{\lambda}{1!}+\ldots+\frac{\lambda^s}{s!}\right]}{\left[1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\ldots+\frac{\lambda^s}{s!}\right]^2+\frac{2\lambda^{s+1}}{(s+1)!}\left[1+\frac{\lambda}{1!}+\ldots+\frac{\lambda^s}{s!}\right]+\left[\frac{\lambda^{s+1}}{(s+1)!}\right]^2}.$$

*To simplify the notation the subscript i has been suppressed.

This will be less than unity if

$$\frac{\lambda^{s+1}}{(s+1)!}\left[1+\frac{\lambda}{s+2}\right]\left[1+\frac{\lambda}{1!}+\ldots+\frac{\lambda^{s}}{s!}\right]<\frac{2\lambda^{s+1}}{(s+1)!}\left[1+\frac{\lambda}{1!}+\ldots+\frac{\lambda^{s}}{s!}\right]+\left[\frac{\lambda^{s+1}}{(s+1)!}\right]^{2},$$

i.e., if

$$\left[\frac{\lambda}{s+2}-1\right]\left[1+\frac{\lambda}{1!}+\ldots+\frac{\lambda^{2}}{s!}\right]<\frac{\lambda^{s+1}}{(s+1)!}. \tag{A.1}$$

Now, the left hand side is

$$-1-\lambda\left[1-\frac{1}{s+2}\right]-\lambda^{2}\left[\frac{1}{2!}-\frac{1}{(s+2)1!}\right]-\lambda^{3}\left[\frac{1}{3!}-\frac{1}{(s+2)2!}\right]$$

$$-\ldots-\lambda^{i}\left[\frac{1}{i!}-\frac{1}{(s+2)(i-1)!}\right]-\ldots-\lambda^{s}\left[\frac{1}{s!}-\frac{1}{(s+2)(s-1)!}\right]$$

$$+\frac{\lambda^{s+1}}{(s+2)s!}.$$

The very last term is less than the right hand side of Equation (A.1). Also, all of the other terms are negative. Therefore, we have proved the inequality (A.1) which completes the proof of concavity.

H

From Equation (19) it is seen that the portion of H which depends upon s_i is

$$\ln\left(\sum_{k=0}^{s_i}\lambda_i^k/k!\right)-\gamma s_i.$$

It is clear that the second difference of this function must be identical to the second difference of $h_i(s_i)$ because γs_i disappears under double differencing. Therefore, if $h_i(s_i)$ is concave, so must be H as a function of s_i.

APPENDIX B—PROOF OF A SPECIAL PROPERTY OF THE OPTIMAL SOLUTION

We consider situations where two conditions are satisfied:

i) $a_i=1 \qquad i=1,2,\ldots,n$

ii) $v_o\leq\sum_{i=1}^{n}v_i.$

From Equations (2), (4), and (9) we have that the ready rate is given by

$$\alpha(s_o,s_1,\ldots,s_n)=C\sum_{x=0}^{s_o}\frac{\lambda_o^{s_o-x}}{(s_o-x)!}\prod_{i=1}^{n}\left(\sum_{k=0}^{s_i+x}\frac{\lambda_i^k}{k!}\right), \tag{B.1}$$

where $C=\exp\left(-\sum_{i=0}^{n} \lambda_i\right)$ does not depend upon the s's.

Suppose all of $s_1, s_2, \ldots, s_n$ are greater than zero. Then another feasible solution with no more budget used (because of condition ii) is $s_0+1, s_1-1, s_2-1, \ldots, s_n-1$. The resulting ready rate is

$$\alpha(s_o+1, s_1-1, \ldots, s_n-1)=C\sum_{x=0}^{s_o+1} \frac{\lambda_o^{s_o+1-x}}{(s_o+1-x)!}\prod_{i=1}^{n}\left(\sum_{k=0}^{s_i-1+x}\frac{\lambda_i^k}{k!}\right).$$

Substituting $m=x-1$ results in

$$\text{(B.2)}\qquad \alpha(s_o+1, s_1-1, \ldots, s_n-1)=C\sum_{m=-1}^{s_o} \frac{\lambda_o^{s_o-m}}{(s_o-m)!}\prod_{i=1}^{n}\left(\sum_{k=0}^{s_i+m}\frac{\lambda_i^k}{k!}\right).$$

It is seen that Equation (B.2) includes all the terms of Equation (B.1) plus the additional positive term (provided $\lambda_o > 0$)

$$C\,\frac{\lambda_o^{s_o+1}}{(s_o+1)!}\prod_{i=1}^{n}\left(\sum_{k=0}^{s_i-1}\frac{\lambda_i^k}{k!}\right).$$

Therefore,

$$\alpha(s_o+1, s_1-1, \ldots, s_n=1) > \alpha(s_o, s_1, \ldots, s_n).$$

In other words, if we have a solution with $s_i>0$ for all $i=1, 2, \ldots, n$ we can do better by reducing all s_1 $(i=1, 2, \ldots, n)$ by one and adding an extra total assembly. In continuing this reasoning we must have at least one s_i $(i>0)$ equal to zero in the optimal solution.

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TERMINATION POLICIES FOR A TWO-STATE STOCHASTIC PROCESS

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ABSTRACT

The purpose of this paper is to investigate and optimize policies which can be used to terminate a two-state stochastic process with a random lifetime. Such a policy consists of a schedule of times at which termination attempts should be made. Conditions are given which reduce the difficulty of finding the optimal policy by eliminating constraints and some boundary points from consideration. Finally, a bound for the optimal policy is derived for a case where some restrictions are imposed on the model.

I. INTRODUCTION

The purpose of this paper is to analyze policies for terminating a costly two-state stochastic process that has a random lifetime. The problem may be easier to visualize if prior to a more detailed description, an application is considered.

Suppose a company has some complex machinery or computing equipment for sale. The distribution of the time it takes to sell the item is known. While on hand the item may suffer a failure, and such an event is costly to the company. The distribution for the time to failure is known; however, the state of the equipment is only revealed when it is tested after its sale. The testing procedure is very expensive and therefore is not applied repeatedly prior to the sale of the item. The company has two alternatives. One is to do nothing and wait until the item is sold. The second choice is to attempt to sell the item with the help of a costly series of advertisements. The probability of success of one such advertisement is known. The problem is to schedule the advertising campaign in order to minimize the costs associated with the disposal of the item.

Another application might be the determination of interception policies for a patrol unit. In this case the patrol has to obtain a schedule of interception attempts against an intruder before he commits some clandestine act.

Areas which seem to be most closely related to the problem are covered in the literature on "Surveillance" and "Optimal Checking Procedures." Surveillance problems were discussed by Savage [8–11], Derman [4] and Roeloffs [7]. Mathematical models for checking procedures include works by Barlow et al. [1], Colemen [3] and Morey [6]. The object of these papers and the topic at hand is to minimize a cost function by finding the best sequence of times for actions relating to a two state process. There are some basic differences between these subjects and termination policies. In surveillance and checking problems the outcome of an inspection depends on the state of the underlying process, and following an inspection the process is repaired and the cycle starts again. In termination problems there is no state dependence, and no recycling occurs following a successful termination attempt. Thus, while the termination problem is not a direct extension of the surveillance problems, the two have a common basis.

II. MODEL FOR TERMINATION POLICIES

Consider a stochastic process which, if unimpeded, is in operation until some random time t, called the "natural termination time." Let $f(t)$ be the probability density function associated with the natural termination time. The process may also be terminated at any time through a successful termination attempt. The probability that the ith such attempt is successful is $p_S(i)$. It is assumed in this paper that $\{p_S(i)\}$ is a monotone sequence defined for all i.

During its existence, the process may be in one of two states—normal or failed. The state of the process is unknown, and can be observed only after termination. Let $h(u|t)du$ be the conditional probability that a transition from the normal to the failed state occurs in the interval $(u, u+du)$, given that termination occurs at time t. The following conditions will be imposed on $h(u|t)$ in this paper:

(a) Failure is not necessarily a certain event, and thus $h(u|t)$ is not a density function;

(b) Prior to termination, the probability of a failure is independent of the time of termination. It is assumed that $h(u|t)$ has the form

$$h(u|t) = \begin{cases} h(u) & u < t \\ 0 & u \geqslant t. \end{cases} \tag{1}$$

Suppose a cost structure is associated with the problem:

$C_N(t) =$ cost if the process is terminated in the normal state at time t.

$C_F(u,t) =$ cost if the process is terminated at time t following a failure which occurred at $u \leqslant t$.

It is assumed that $C_N(t) \leqslant C_F(u,t)$ and that both costs are nondecreasing functions of t, and that $C_F(u,t)$ is also a nondecreasing function of $t-u$, the time spent in the failed state.

Suppose at time $t_0 = 0$ a termination policy is invoked. Let this policy be denoted by $T_P = \{t_1, t_2, \ldots\}$ where t_i is the time the ith termination attempt is made. Let the cost structure of the attempts be defined as

$C_A(t) =$ cost of a termination attempt made at time t; assumed to be nonincreasing function of t.

It is further assumed that all probability densities and cost functions have finite first and second derivatives in their interval of definition.

III. EXPECTED COST OF A POLICY

The object of this paper is to analyze these simplified termination policies so as to minimize the total expected cost of the process and the termination attempts. Let the expected cost of the unimpeded process up to a natural termination time of t be $C_T(t)$. The expectation is taken with respect to the time of failure. $C_T(t)$ may be written as

$$C_T(t) = \int_0^t C_F(u,t)h(u)du + \left[1 - \int_0^t h(u)du\right] C_N(t) \cdot \tag{2}$$

Using the appropriate definitions of the cost functions involved it can be shown that $C_T(t)$ is a non-decreasing function of the time t. In addition, it is assumed that the expected value of $C_T(t)$ is finite.

Suppose k termination attempts are scheduled. Then, the expected total cost associated with a policy $T_P = \{t_1, t_2, \ldots, t_k\}$ may be shown to be

$$(3) \qquad C_P(t_1,t_2,\ldots,t_k) = \int_0^\infty C_T(t)f(t)dt - \sum_{i=1}^{k} P_{\bar{S}}(i-1)p_S(i)\int_{t_i}^\infty [C_T(t)-C_T(t_i)]f(t)dt + \sum_{i=1}^{k} P_{\bar{S}}(i-1)G(t_i)C_A(t_i),$$

where

$$(4) \qquad P_S(i-1) = \begin{cases} \prod_{j=1}^{i-1}(1-p_S(j)) & i=2,3,\ldots \\ 1 & i=1 \end{cases}$$

and

$$(5) \qquad G(t_i) = \int_{t_i}^\infty f(t)dt.$$

Since scheduling an attempt for infinity is equivalent to not scheduling the attempt at all, it may be assumed that $k=\infty$. This still allows for having only a finite number of attempts, n, to be scheduled by letting $t_i=\infty$ for i greater than n. It will be assumed that if an infinite number of attempts are made the probability of never succeeding goes to zero, i.e., $\lim_{k\to\infty} P_{\bar{S}}(k)=0$ and the expected number of attempts, $\bar{N}$, necessary for success if finite. In the remaining portion of this paper k will be set equal to ∞.

For the case where $k=\infty$, Equation (3) may be rewritten in a shortened form as

$$(6) \qquad C_P(t_1,t_2,\ldots) = \int_0^\infty C_T(t)f(t)dt + \sum_{i=1}^{\infty} C_i(t_i),$$

where

$$(7) \qquad C_i(t_i) = P_{\bar{S}}(i-1)L_i(t_i),$$

$$(8) \qquad L_i(t_i) = G(t_i)C_A(t_i) - p_s(i)I(t_i),$$

and

$$(9) \qquad I(t_i) = \int_{t_i}^\infty [C_T(x)-C_T(t_i)]f(x)dx.$$

An optimal termination policy is one which minimizes $C_P(t_1, t_2, \ldots)$. Let $T_P^* = \{t_1^*, t_2^*, \ldots\}$ be such a policy. Then, since the expected value of $C_T(t)$ is assumed finite, T_P^* satisfies

(10) $$\sum_{i=1}^{\infty} C_i(t_i^*) = \min_{t_1 \leqslant t_2 \leqslant \cdots} \sum_{i=1}^{\infty} C_i(t_i).$$

Notice that when all the attempts are scheduled for infinity, i.e., $t_i = \infty$, $i = 1, 2, \ldots$, then $C_i(t_i) = 0$ for all i. This means that the optimal policy has an upperbound, i.e.;

(11) $$\sum_{i=1}^{\infty} C_i(t_i^*) \leqslant 0.$$

IV. PROPERTIES OF THE GLOBAL MINIMA

Now two theorems about the properties of the point where the total cost is minimized may be proved.

THEOREM 1: For every i let $\hat{t}_i$ be a point such that $C_i(\hat{t}_i) \leqslant C_i(t)$ for all $t \geqslant 0$. If $p_S(j) \geqslant p_S(k)$, then there exists a set of points t'_i, such that $C_i(t'_i) = C_i(\hat{t}_i)$ and $t'_j \leqslant t'_k$.

PROOF: Let $\Delta(t)$ be defined as

(12) $$\Delta(t) = L_k(t) - L_j(t) = I(t)[p_S(j) - p_S(k)].$$

Notice that since $I(t) \geqslant 0$ and $p_S(j) \geqslant p_S(k)$, $\Delta(t) \geqslant 0$. Also it can be seen from Equation (9) that $I(t)$ varies inversely with t and thus $\Delta(t)$ also varies inversely with t. Let $L_k(t)$ be rewritten as

(13) $$L_k(t) = \Delta(t) + L_j(t).$$

Consider Equation (19) for all $t \leqslant t'_j$; t'_j is any point such that $C_j(\hat{t}) = C_j(t')$. Since $C_j(t'_j) \leqslant C_j(t)$ for all $t \geqslant 0$, from (7) it is also true that $L_j(t'_j) \leqslant L_j(t)$ for all $t \geqslant 0$. Also, since $\Delta(t)$ varies inversely with t the inequality $\Delta(t'_j) \leqslant \Delta(t)$ must hold for all $t \leqslant t'_j$. Now, Equation (13) can be transformed into the following inequality

(14) $$L_k(t) \geqslant \Delta(t'_j) + L_j(t'_j) = L_k(t'_j) \qquad \text{for } t \leqslant t'_j.$$

Since the absolute minimum value of $C_k(t)$ and thus $L_k(t)$ occurs at $t = \hat{t}_k$, $L_k(\hat{t}_k) \leqslant L_k(t)$ for $t \geqslant 0$. Now the required solution can be constructed as follows:

If

$$L_k(\hat{t}_k) < L_k(t'_j) \qquad \text{then let } t'_k = \hat{t}_k$$

and if

(15) $$L_k(\hat{t}_k) = L_k(t'_j) \qquad \text{then let } t'_k = t'_j.$$

From (14) and (15) it is clear that the absolute minimum of $L_k(t)$ and thus $C_k(t)$ occurs at a point $t'_k \geqslant t'_j$. Q.E.D.

From the above theorem it is clear that if $\{p_s(i)\}$ is a monotonically decreasing sequence, then the points of absolute minima t_i^*, $i=1, 2, \ldots$ will satisfy the constraint set $\{t_i \leq t_{i+1};\ i=1, 2, \ldots\}$ automatically.

THEOREM 2: Let $\{p_s(i)\}$ be a nondecreasing sequence. Then, the optimal solution to the termination problem will be such that $t_1^*=t_2^*= \ldots$.

PROOF: Let t_i^*, $i=1, 2, \ldots$ be an optimal solution. Then, it must be true that $t_i^* \leq t_{i+1}^*$ and

$$\text{(16)} \qquad \text{or} \qquad \left.\begin{array}{l} C_i(t_i^*) \leq C_i(t) \\ \\ L_i(t_i^*) \leq L_i(t) \end{array}\right\} t_i^* \leq t \leq t_{i+1}^*.$$

From (16) it can be seen that

$$\text{(17)} \qquad L_{i+1}(t_i^*) \geq L_{i+1}(t_{i+1}^*).$$

From (16)

$$\text{(18)} \qquad L_i(t_{i+1}^*) \geq L_i(t_i^*).$$

Combining (17) and (18) and rearranging terms yield

$$\text{(19)} \qquad L_{i+1}(t_i^*) - L_i(t_i^*) \geq L_{i+1}(t_{i+1}^*) - L_i(t_{i+1}^*).$$

By use of Equation (18), the above implies that

$$-I(t_i^*)[p_s(i+1)-p_s(i)] \geq -I(t_{i+1}^*)[p_s(i+1)-p_s(i)]$$

or

$$\text{(20)} \qquad I(t_i^*) \leq I(t_{i+1}^*).$$

Since $I(t)$ varies inversely with t (20) implies that $t_i^* \geq t_{i+1}^*$. Combining this with the feasibility condition $t_i^* \leq t_{i+1}^*$ one obtains that $t_i^*=t_{i+1}^*$. Q.E.D.

The results of the above theorems may be summarized as follows:

(i) For $\{p_S(i)\}$ nonincreasing there exists an optimal policy, $T_P^*=\{t_1^*, t_2^*, \ldots\}$, where t_i^* is such that

$$C_i(t_i^*) = \min_{t_i \geq 0} C_i(t_i) \qquad i=1,2, \ldots.$$

(ii) For $\{p_S(i)\}$ nondecreasing the optimal policy is $T_P^*=\{t^*, t^*, \ldots\}$, where t^* is such that

$$\sum_{i=1}^{\infty} C_i(t^*) = \min_{t \geq 0} \sum_{i=1}^{\infty} C_i(t).$$

Notice that in either case the problem has been reduced by removing the ordering constraints on the t_i's. The sections that follow will be devoted to characterizing the solutions to such problems.

V. MINIMIZING THE EXPECTED COST

To find the optimal solution when $\{p_S(i)\}$ is a nonincreasing sequence the minimum of $C_i(t_i)$ may be found using the usual classical techniques. In order to facilitate the analysis the integral $I(t)$, of Equation (9) will be rewritten.

When it is assumed that

(i) $C_T(t)$ is differentiable everywhere with derivative $C'_T(t)$, and

(ii) $\lim_{t\to\infty} C_T(t)\, G(t)=0$,

$$I(t_i)=\int_{t_i}^{\infty} C'_T(t)G(t)\,dt. \tag{21}$$

From Equations (7), (8), and (21) the first derivative of $C_i(t_i)$ is

$$\frac{dC_i(t_i)}{dt_i}=P_S(i-1)\,\{-C_A(t_i)f(t_i)+C'_A(t_i)G(t_i)+p_S(i)C'_T(t_i)G(t_i)\}. \tag{22}$$

If the above derivative is set equal to zero the following critical equation results.

$$-C_A(t_i)r(t_i)+C'_A(t_i)+p_S(i)C'_T(t_i)=0, \tag{23}$$

where

$$r(t_i)=\frac{f(t_i)}{G(t_i)}. \tag{24}$$

Let $t_i=\tau_i$ be the point where Equation (23) is satisfied. Then $C_i(t_i)$ will have a relative minimum at $t_i=\tau_i$ if the second derivative at this point is positive.

Under certain conditions the series of theorems that follow can be used to eliminate either the points of relative minima or the boundaries from consideration in the search for the global minimum.

THEOREM 3: If $C_i(t_i)$ has a relative minimum at $t_i=\tau_i$ and if

$$\left.\frac{d^2C_i(t_i)}{dt_i^2}\right|_{t_i=\tau_i} > 0, \text{ then}$$

$$\frac{d\tau_i}{dp_S(i)}\leq 0.$$

PROOF: Consider the critical equation, Equation (23). Taking the derivative using the chain rule one obtains

$$\frac{d}{d\tau_i}\left[-C_A(\tau_i)r(\tau_i)+C'_A(\tau_i)+p_S(i)C'_T(\tau_i)\right]\,\frac{d\tau_i}{dp_S(i)}+C'_T(\tau_i)=0.$$

Solving for $\frac{d\tau_i}{dp_S(i)}$ yields

$$\frac{d\tau_i}{dp_S(i)}=\frac{-C_T'(\tau_i)}{\frac{d}{d\tau_i}\{-C_A(\tau_i)r(\tau_i)+C_A'(\tau_i)+p_S(i)C_T'(\tau_i)\}}.$$

At a relative minimum the denominator is positive as the derivative of Equation (22) is positive, and since $C_T(t)$ is a nondecreasing function the numerator is nonpositive. Thus, the ratio is nonpositive. Q.E.D.

THEOREM 4: If $C_i(t_i)$ has a relative minimum at $t_i=\tau_i$, and

$$\left.\frac{d^2C_i(t_i)}{dt_i}\right|_{t_i=\tau_i}>0,$$

then $C_i(\tau_i)$ is a nonincreasing concave function of $p_S(i)$.

PROOF: The first derivative of $C_i(\tau_i)$ with respect to $p_S(i)$ reduces to

$$\frac{dC_i(\tau_i)}{dp_S(i)}=-\left(\int_{\tau_i}^{\infty}C_T'(u)G(u)du\right)P_{\bar{S}}(i-1)\leqslant 0. \tag{25}$$

The second derivative is

$$\frac{d^2C_i(\tau_i)}{dp_S(i)}=C_T'(\tau_i)G(\tau_i)\,\frac{d\tau_i}{dp_S(i)}\,P_{\bar{S}}(i-1). \tag{26}$$

It has been shown that

$$\frac{d\tau_i}{dp_S(i)}\leqslant 0$$

and since $C_T'(\tau_i)\geqslant 0$ the second derivative is $\leqslant 0$, implying that $C_i(\tau_i)$ is a concave function. Q.E.D.

Theorem 4 can be used to narrow the search for a minimum.

THEOREM 5: Let $p_S(j+1)\leqslant p_S(j)$, and suppose τ_j and τ_{j+1} are points such that the first derivative of $C_i(t_i)$ vanishes and the second derivative is positive. Then,

$$C_{j+1}(\tau_{j+1})-C_{j+1}(0)\leqslant C_j(\tau_j)-C_j(0). \tag{27}$$

PROOF: The following well known property of concave functions [5] will be used. If $b(x)$ is concave for $x\epsilon X$, then for $x,y\epsilon X$

$$b(y)-b(x)\leqslant\frac{db}{dx}\,(y-x).$$

Consider $C_i(\tau)$ as a function of $p_S(i)$ and τ a point of relative minimum. Applying the above property

and Theorem 4 we get

$$C_{j+1}(\tau_{j+1}) - C_j(\tau_j) \leq -\{p_S(j+1) - p_S(j)\}P_{\bar{S}}(j-1)I(\tau_j). \tag{28}$$

By use of Equations (7) and (8) it can be shown that

$$C_{j+1}(0) - C_j(0) \geq \{p_S(j)(1-p_S(j+1)) - p_S(j+1)\}P_{\bar{S}}(j-1)I(0) \geq \{p_S(j) - p_S(j+1)\}P_{\bar{S}}(j-1)I(0). \tag{29}$$

Subtracting (29) from (28) yields

$$(C_{j+1}(\tau_{j+1}) - C_{j+1}(0)) - (C_j(\tau_j) - C_j(0)) \leq (p_S(j+1) - p_S(j))(I(0) - I(\tau_j)). \tag{30}$$

The right-hand side of (30) is ≤ 0, proving the desired result. Q.E.D.

Corollaries of Theorem 5 can be applied to eliminate some of the boundary points from consideration.

Corollary 5a: If $p_S(i)$ is a nonincreasing function of i, k is an integer such that $C_k(\tau_k) \leq C_k(0)$, and if the assumptions of Theorem 5 hold, then for all $j \geq k$ $C_j(\tau_j) \leq C_j(0)$. In particular if $k=1$ none of the boundary points need be checked as they all yield results which are no better than those resulting from the solution to the critical equation. The corollary is a direct result of Theorem 5.

Corollary 5b: If $p_S(i)$ is a nonincreasing function of i, k is some integer such the $C_k(\tau_k) \geq C_k(0)$, and if the assumptions of Theorem 5 hold, then for all $j \leq k$ $C_j(\tau_j) \geq C_j(0)$. In particular if $\lim_{k\to\infty} [C_k(\tau_k) - C_k(0)] \geq 0$, then none of the critical equations need be solved as the solution at the boundary is always at least as good. This corollary is also a direct result of Theorem 5.

For the case where $\{p_S(i)\}$ is a nondecreasing sequence the minimum of $\sum_{i=1}^{\infty} C_i(t)$ must be found to solve the problem. The theorem below shows that solving for this minimum involves a critical equation of the same form as the one for $C_i(t_i)$.

THEOREM 6: The function $\sum_{i=1}^{\infty} C_i(t)$ has a relative minimum at $t=\tau$ if

$$-r(\tau)C_A(\tau) + C_A'(\tau) + pC_T'(\tau) = 0, \tag{31}$$

where

$$P = \frac{1}{\bar{N}}$$

and $\bar{N}$ is the expected number of attempts needed to be successful.

PROOF: From Equations (7) and (8) $\sum_{i=1}^{\infty} C_i(t)$ can be expressed as:

$$\sum_{i=1}^{\infty} C_i(t) = \sum_{i=1}^{\infty} P_{\bar{S}}(i-1)\,[G(t)C_A(t) - p_S(i)I(t)]. \tag{32}$$

Collecting terms and dividing both sides by $\sum_{i=1}^{\infty} P_{\bar{S}}(i-1)$ yields

(33) $$\frac{1}{\sum_{i=1}^{\infty} P_{\bar{S}}(i-1)} \sum_{i=1}^{\infty} C_i(t) = G(t)C_A(t) - p\int_t^{\infty} C_T'(x)G(x)dx.$$

The right side of Equation (33) has the same form as $C_i(t_i)$ and thus setting its first derivative equal to zero yields the same result as Equation (23) with p replacing $p_S(i)$. Q.E.D.

VI. BOUNDS FOR THE OPTIMA WHEN $C_T(t)$ IS CONVEX

This section contains a theorem which shows that if $C_T'(t)$ is increasing, then a bound for $C_i(t_i)$ can be found. This theorem is derived from results in reliability theory by considering failure rates.

THEOREM 7: Let the natural termination time have density $f(t)$, mean $1/\mu$ and an increasing (decreasing) failure rate, $r(t)$. Let $t_i = \tau_i$ be a point where the critical equation $\frac{dC_i(t_i)}{dt_i} = 0$ is satisfied. Let $C_i^e(t_i)$ represent the cost function, $C_i(t_i)$, when the natural termination density is $\mu e^{-\mu t}(t \geqslant 0)$. If $C_T'(t)$ is increasing ($C_T(t)$ is convex), then

(34) $$C_i(\tau_i) \underset{(\leqslant)}{\geqslant} C_i^e(\tau_i).$$

PROOF: The proof for the case where $r(t)$ is increasing will be given here. The proof for the case where $r(t)$ is decreasing is similar.

To prove Inequality (34), it must be shown that

(35) $$C_A(\tau_i)[G(\tau_i) - e^{-\mu\tau}i] - p_S(i)\int_{\tau_i}^{\infty} C_T'(t)[G(t) - e^{-\mu t}]dt \geqslant 0.$$

If $r(t)$ is increasing, then $G(t)$ crosses $e^{-\mu t}$ from above at most once for $t > 0$ [2]. Let $t = t_0$ be the point where $G(\pm) = e^{-\mu t}$. From Lemma 2 in the appendix

$$\int_{\tau_i}^{\infty} C_T'(t)[G(t) - e^{-\mu t}]dt \leqslant 0.$$

Thus, since for $t \leqslant t_0 G(t) \geqslant e^{-\mu t}$, the inequality clearly holds.

Next, consider the case when $\tau_i > t_0$. Since the critical equation holds at $t_i = \tau_i$, and by assumption $C_A'(\tau_i) \geqslant 0$ for all t_i, from equation (23) and Lemma 1 in the appendix

(36) $$C_A(\tau_i) \leqslant p_S(i)\frac{C_T'(\tau_i)}{r(\tau_i)} \leqslant p_S(i)\frac{C_T'(\tau_i)}{\mu}.$$

Suppose the theorem does not hold. Then, from (35)

$$C_A(\tau_i)[e^{-\mu\tau_i}-G(\tau_i)] > p_S(i)\int_{\tau_i}^{\infty} C_T'(t)[e^{-\mu t}-G(t)]dt. \tag{37}$$

Using (36) and the facts that $C_T'(t)$ is increasing and $e^{-\mu\tau_i}-G(\tau_i)\geqslant 0$ for $\tau_i \geqslant t_0$, we obtain

$$\frac{p_S(i)C_T'(\tau_i)}{\mu}\left[e^{-\mu\tau_i}-G(\tau_i)\right] > p_S(i)C_T'(\tau_i)\int_{\tau_i}^{\infty}[e^{-\mu t}-G(t)]dt$$

or

$$e^{-\mu\tau_i}-G(\tau_i) > \mu\int_{\tau_i}^{\infty}[e^{-\mu t}-G(t)]dt. \tag{38}$$

By integrating the right-hand side the following simple inequality can be derived,

$$0<\int_{\tau_i}^{\infty}\{\mu G(t)-f(t)\}dt. \tag{39}$$

However, for all $t\geqslant\tau_i>t_0$, from Lemma 1 in the appendix it is clear that

$$0\geqslant \mu G(t)-f(t) \tag{40}$$

contradict, Inequality (39).

Note, that using the inequalities of the previous theorem the following general conclusion can be made about the problem. If the failure rate of natural termination is increasing (decreasing), then at the absolute minimum, τ_i^*, $C_i^e(\tau_i^*)\ \underset{(\geqslant)}{\leqslant}\ C_i(\tau_i^*)$. This is the direct result of the fact that at $t=\infty$ $C_i^e(t)=C_i(t)=0$ and at $t=0$ or $t=\tau_i$, $C_i^e(t)\ \underset{(\geqslant)}{\leqslant}\ C_i(t)$ by the previous theorem.

VII. SUMMARY

The goal of this paper has been to analyze policies which can be used to terminate a two-state stochastic process having a random lifetime. A termination policy may be denoted by $T_P=(t_1, t_2, \ldots)$, where t_i is the time for the ith attempt. It has been shown that the optimal policy reduces to the simple forms listed below when the probability of success on the ith termination attempt, $p_S(i)$, is a monotonic function of i.

(1) If $\{p_S(i)\}$ is nonincreasing the optimal policy, is $T_P^*=(t_1^*, t_2^*, \ldots)$, where t_i^* is the point at which the cost function $C_i(t_i)$ achieves its minimum.

(2) If $\{p_S(i)\}$ is nondecreasing the optimal policy is $T_P^*=(t^*, t^*, \ldots)$, where t^* is such that

$$\sum_{i=1}^{\infty} C_i(t^*)=\min_{t\geqslant 0}\sum_{i=1}^{\infty} C_i(t).$$

Simple conditions, under which either the boundary points ($t_i=0$), or relative minima could be eliminated from consideration have been derived. Finally, under special assumptions a bound on the optimal cost has been found.

The research on the termination problem may be extended in several directions. For example the assumption that $\{p_S(i)\}$ is a monotone sequence can be removed. In that case a new method for computing the feasible minima need be considered. It may be assumed that the probability of success is a function of time, $p_S(t_i)$. The objective function of such a problem would not be separable, necessitating a different method of solution. Other extensions to the problem may result from consideration of specific cost and distribution functions. These can lead to closed form solutions which may then be applied to real problems.

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Appendix 1

PROOF OF LEMMAS 1 AND 2

LEMMA 1: Suppose $r(t)=\dfrac{f(t)}{G(t)}$ is an increasing (decreasing) failure rate and $G(t) \underset{(\geq)}{\leq} e^{-\mu t}$ for all $t > t_0$. Then, $r(t) \underset{(\leq)}{\geq} \mu$ for all $t > t_0$.

PROOF: Let $G(t) \leq e^{-\mu t}$ for $t \geq t_0$. From reliability theory [2] the above can also be written as

$$\exp\left[-\int_0^t r(x)dx\right] \leq \exp\left[-\int_0^t \mu dx\right].$$

Taking the logarithm of both sides and combining terms yields

$$\int_0^t (r(x)-\mu)dx \geq 0.$$

In particular, if $t=t_0$ the above inequality will only hold if $r(x) \geq \mu$ for some $x < t_0$. But since $r(x)$ is increasing $r(x) \geq \mu$ for all $x > t_0$. The proof for the case when $r(t)$ is decreasing is similar and is not given here.

LEMMA 2: Let $f(x)$ be a density with mean $1/\mu$ whose failure rate $r(x)$ is increasing (decreasing). Let $\phi(t)$ be a nonnegative, increasing function of t. Then for all $t \geq 0$

$$\int_t^\infty \phi(x)G(x)dx \underset{(\geq)}{\leq} \int_t^\infty \phi(x)e^{-\mu x}dx.$$

PROOF: In Reference [2] the inequality has been shown for the case where $t=0$. Suppose $r(x)$ is increasing, then $G(x)=\int_x^\infty f(u)du$ crosses $e^{-\mu x}$ from above at some point $x=x_0$ [2]. Thus for

$x \geqslant x_0 G(x) \leqslant e^{-\mu x}$ and thus

$$\int_t^\infty \phi(x)[G(x) - e^{-\mu x}] \leqslant 0$$

and the lemma holds for $t \geqslant x_0$. Suppose $t < x_0$. Since the lemma holds at zero, we can write

$$\int_0^\infty \Phi(x)[G(x) - e^{-\mu x}]dx \leqslant 0.$$

Equivalently, it is also true that

$$\int_0^t \phi(x)[G(x) - e^{-\mu x}]dx + \int^\infty \phi(x)[G(x) - e^{-\mu x}]dx \leqslant 0$$

or

$$\int_0^t \phi(x)[G(x) - e^{-\mu x}]dx \leqslant -\int_t^\infty \phi(x)[G(x) - e^{-\mu x}]dx.$$

But for $x < t \leqslant x_0 G(x) > e^{-\mu x}$, thus the left hand side of the inequality is nonnegative or

$$\int_t^\infty \phi(x)[G(x) - e^{-\mu x}]dx \leqslant 0.$$

Q.E.D.

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COMPUTATIONAL TECHNIQUES FOR OPTIMIZING SYSTEMS WITH STANDBY REDUNDANCY

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ABSTRACT

Three methods are used to solve the following problem: For P, a positive constant, maximize (P. Reliability-cost) of a system with standby redundancy. The results show that a method which rounds a noninteger solution to the nearest integer solution can lead to tremendous mistakes. However, neither a well known dynamic programming algorithm nor a previously developed branch and bound technique are able to solve large size problems. The solution of problems of large dimension thus requires the use of the noninteger solution of the first method to limit the number of possible solutions when using either the dynamic programming algorithm or a modified branch and bound technique. With this assistance, the branch and bound technique is able to solve large problems in a short amount of computational time.

1. INTRODUCTION

During the past 10 years, a variety of methods and algorithms have been proposed for solving problems involving standby redundancy in serial systems. In this paper, we present a comparison of the computational effectiveness of three methods applied to a specific problem of standby redundancy.

The first method uses control theory to get a noninteger optimal solution to the problem which is then rounded to the nearest integer solution [1]. The second method uses Ketelle's algorithm [5] to compute the solution from among the terms of an undominated optimal sequence. The third method modifies a branch and bound technique developed in [4] for solving a different reliability problem from that considered in this paper. These three methods will first be applied without modification. Each is then improved by the use of theorems, given in the following section, concerning the form of the optimal solution. The modified branch and bound procedure seems especially promising for solving large problems in very short time.

From these experiments it appears that rounding to the nearest integer solution is a convenient way for approximating the optimal solution, but can sometimes lead to important errors. However, the noninteger solution can be used for limiting the number of possible integer solutions which can be considered in other algorithms. We also show that, for the problem treated in this paper, the use of a discrete maximum principle is redundant. The noninteger solution can be obtained directly by setting the first derivatives of the objective function equal to zero.

2. THE PROBLEM AND ITS STRUCTURE

We consider a system of n stages in series. Each stage uses a single component which can be replaced by standby units in a loaded mode.* A component at stage $i (i = 1, \ldots, n)$ has a probability p_i of failing before the completion of the mission. If there are $(m_i - 1)$ standbys at stage $i (i = 1, \ldots, n)$ the reliability of the system is: $R = \prod_{i=1}^{n} (1 - p_i^{m_i})$. Components for stage i cost $c_i (i = 1, \ldots, n)$, so that the total cost $C = \sum_{i=1}^{n} m_i c_i$ (see [4]).

If we associate a revenue P with the successful completion of the mission, the expected profit becomes $PR - C = P \left[\prod_{i=1}^{n} (1 - p_i^{m_i}) \right] - \sum_{i=1}^{n} m_i c_i$. The problem is then to maximize this expected profit; i.e., to determine the m_i's which maximize the objective function $(PR - C)$. This problem will be solved by three different methods.

Other methods have been proposed (see [3] and [6]) for solving more or less similar problems. In [3] for example, Everett uses generalized Lagrange multipliers to generate optimal solutions to the problem of maximizing reliability subject to certain cost constraints. In our problem, we could consider $1/P$ as the special value of the Lagrange multiplier for which R is maximized. But by this way we would not maximize the function $(R - C/P)$ which is our objective. This explains why the results and techniques of this paper are different from those of [3]. Moreover, although our objective function seems to correspond to a special value $1/P$ of the Lagrange multiplier, the actual problem is not separable in each variable as in [3].

First we present theorems which enable us to reduce the search for the optimal solution.

THEOREM I: The (noninteger) values of m_i maximizing the objective function $(PR - C)$ are either 0 or satisfy the following system of equations:

$$P \prod_{i=1}^{n} (1 - y_i) = a_i (1 - y_i)/y_i \; (i = 1, \ldots, n) \text{ with } y_i = p_i^{m_i} \quad \text{and } a_i = - c_i / \log (p_i) > 0.$$

PROOF: A maximum of the objective function is either the point $(0, \ldots, 0)$† or a local extremum. For such an extremum, the first derivatives with respect to $m_i (i = 1, \ldots, n)$ must be equal to zero. So the conditions become:

$$\frac{\partial (PR - C)}{\partial m_i} = \frac{PR(-p_i^{m_i}) \log (p_i)}{(1 - p_i^{m_i})} - c_i = 0 \qquad (i = 1, \ldots, n) \text{ to prove the theorem.}$$

LEMMA I (Theorem 4.6 in [8]): For any convex function f and any α belonging to $[-\infty, +\infty]$, the level sets $\{x : f(x) < \alpha\}$ and $\{x : f(x) \leq \alpha\}$ are convex.

LEMMA II: The set of points $m_i (i = 1, \ldots, n)$, such that $\sum_{i=1}^{n} p_i^{m_i} \leq 1$ is a convex set.

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*A series-parallel arrangement where the stages are parallel subsystems, is a particular example of a series system with loaded standbys. But more generally, there exist systems with standbys such that the standbys have the same failure law as the basic units.

†Any other point on the frontier has a reliability of 0, cost something, and therefore its objective function is negative.

PROOF: The function $\sum_{i=1}^{n} p_i^{m_i}$ is convex. This is evident, because the Hessian matrix is a diagonal matrix whose terms on the main diagonal are equal to $p_i^{m_i}[\log(p_i)]^2$. Thus the principal minors are positive. Then, it is sufficient to apply Lemma I to prove this Lemma.

THEOREM II: The function $(PR-C)$ is concave on the set of m_i's such that $\sum_{i=1}^{n} p_i^{m_i} \leq 1$.

PROOF: (a) The corresponding Hessian matrix is:

$$H_{ij}=\begin{cases} PR\,\dfrac{p_i^{m_i}p_j^{m_j}\log(p_i)\log(p_j)}{(1-p_i^{m_i})(1-p_j^{m_j})} & \text{for } i\neq j \\[2ex] -PR\,\dfrac{p_i^{m_i}[\log(p_i)]^2}{(1-p_i^{m_i})} & \text{for } i=j. \end{cases}$$

In order for the function to be concave, the principal minors of the Hessian matrix must alternate in signs; i.e., for the principal minor of order k, $|H_k|$, the condition is:

$$\operatorname{sign}|H_k|=\operatorname{sign}\begin{vmatrix} H_{11} & \cdots & H_{1k} \\ \vdots & & \\ H_{k1} & & H_{kk}\end{vmatrix} \begin{matrix} =0 \\ \text{or} \\ (-1)^k \end{matrix} \qquad (k=1,\ldots n).$$

In order to compute sign $|H_k|$, we can factor out some common positive variables and write:

$$|H_k|=\frac{(PR)^k\left[\prod_{i=1}^{k}[\log(p_i)]^2\right]|h_k|}{\left[\prod_{i=1}^{k}(1-p_i^{m_i})\right]^2} \qquad \text{so that sign } |H_k|=\text{sign } |h_k|$$

with

$$|h_k|=\begin{vmatrix} -(1-p_1^{m_1}) & p_2^{m_2} & \cdots\cdots & p_k^{m_k} \\ p_1^{m_1} & (1-p_2^{m_2}) & \cdots\cdots & p_k^{m_k} \\ \vdots & \vdots & & \vdots \\ p_1^{m_1} & p_2^{m_2} & & (1-p_k^{m_k}) \end{vmatrix}$$

Subtracting the first row from each of the other rows yields:

$$|h_k|=\left|\begin{array}{c|ccc} p_1^{m_1}-1 & p_2^{m_2} & \cdots & p_k^{m_k} \\ \hline 1 & & & \\ \vdots & & -I_{k-1} & \\ 1 & & & \end{array}\right| \qquad (I_j \text{ being the unit matrix of order } j),$$

and adding column 2 through k to the first column:

$$|h_k| = \left|\begin{array}{c|cccc} \sum_i p_i^{m_i} - 1 & p_2^{m_2} & \cdots & \cdots & p_k^{m_k} \\ \hline 0 & -I_{k-1} & & & \end{array}\right| .$$

Therefore, if the condition on H_n is satisfied, so are the conditions on the other principal minors. Thus the objective function is concave at any point $\{m_i\}$ such that $\sum_{i=1}^{n} p_i^{m_i} \leqslant 1$.

(b) From Lemma II, we know that the set of such points is convex.

(c) Therefore, the function $(PR - C)$ being locally concave at any point of the convex set $\left\{\sum_{i=1}^{n} p_i^{m_i} \leqslant 1\right\}$ is concave in that set, and the theorem has been proven.

THEOREM III: The system of equations

$$(1) \qquad P \prod_{i=1}^{n} (1 - y_i) = a_1(1 - y_1)/y_1 = \ldots = a_n(1 - y_n)/y_n (0 < y_i < 1) \qquad \text{for all } i\text{'s})$$

admits at most two solutions.

PROOF: (a) From the last $(n - 1)$ equations, $y_i (i = 2, \ldots n)$ can be expressed as an increasing function of y_1. To each value of y_1 corresponds one and only one value of $y_i (i = 2, \ldots, n)$:

$$y_i = \{1 + (a_1/a_i)[(1 - y_1)/y_1]\}^{-1}$$

which after some transformations gives:

$$y_i = 1 - (1 - y_i)/[1 + (a_i/a_1 - 1)y_1].$$

We can always suppose that the stages are ordered such that $a_j \geqslant a_{j-1} (j=1, \ldots, n)$. Therefore, $A_i = a_i/a_i - 1 \geqslant 0$ and y_i is an increasing function of y_i.

(b) We now replace $(1 - y_i)$ in the first equation of (1) as functions of y_1 to get:

$$P(1 - y_1)^n / \left[\prod_{i=2}^{n} (1 + A_i y_1)\right] = a_1(1 - y_1)/y_1,$$

or

$$P a_1^{-1} y_1 (1 - y_1)^{n-1} = \prod_{i=2}^{n} (1 + A_i y_1).$$

For simplifying, let us call $LS(y_1)$, the expression $P a_1^{-1} y_1 (1 - y_1)^{n-1}$ and $RS(y_1)$ the expression $\prod_{i=2}^{n} (1 + A_i y_1)$. $LS(y_1)$ is a unimodal function of y_1, concave up to a certain value of y_1, y_1^*, and convex after this value. For this value y_1^*, $LS(y_1)$ is decreasing. $RS(y_1)$ is a nondecreasing convex function of y_1. Therefore, we have one of the following situations:

(a) $LS(y_1)$ and $RS(y_1)$ have no intersection point.

(b) The smallest intersection point $\bar{y}_1$ is such that $LS(\bar{y}_1)$ is increasing since $LS(0)=0$ and $RS(0) >=1$. Then $LS(y_1)$ and $RS(y_1)$, must have exactly one more intersection point (except in case of tangency between the two curves at $\bar{y}_1$) by using the above properties of $LS(\cdot)RS(\cdot)$.

THEOREM IV: The function $(OR-C)$ admits at most one local maximum, located in the region $\sum_{i=1}^{n} p_i^{m_i} \leq 1$.

PROOF: In order to have a local maximum, the system of Equations (1) must be satisfied. From the former theorem, only two points can satisfy these equations. On the other side, a maximum must correspond to alternate principal minors of the Hessian matrix. Therefore a maximum must be in the region $\sum_{i=1}^{n} p_i^{m_i} \leq 1$. The function $(PR-C)$ is concave in that region. So it is impossible to have two local maxima inside this region. (Otherwise the function would be larger at all points between these two maxima, which is impossible.) Moreover, if two points satisfy Equations (1), the maximum, corresponds to the points with the largest m_i† $(i=1, \ldots, n)$ if it is such that $\sum_{i=1}^{n} p_i^{m_i} \leq 1$.

THEOREM V: An optimal solution $m_i^*(i=1, \ldots, n)$ (m_i^* integer) is such that $m_i^* \leq M_i$, where $M_i = \min\{k\colon (p_i^k - p_i^{k+1}) \leq c_i/P\}$.

PROOF: Suppose that, at the optimal solution, there exists a stage j such that $m_j^* = M_j + m$ (m a positive integer). Then, we show that by replacing m_j^* by M_j, we improve the objective function to arrive at a contradiction.

Let R_j represent the reliability of the whole system, stage j expected, at the optimal solution. The reliability of our optimal solution is: $R_j(1-p_j^{m_j^*})$. Therefore, by replacing m_j^* by M in the objective function, we increase it by:

$$mc_j - PR_j(p_j^{M_j} - p_j^{M_j+m}) \geqslant mc_j - mPR_j(p_j^{M_j} - p_j^{M_j+1})\dagger\dagger \geqslant mc_j - mP(p_j^{M_j} - p_j^{M_j+1})$$

$$\geqslant mc_j - mc_j = 0, \qquad \text{which proves the theorem.}$$

Let R and C be the optimal reliability and cost corresponding to a value P and R', C' those corresponding to a value $P' > P$.

THEOREM VI:

(a) If $P' \geqslant P$, then $R' \geqslant R$.

(b) The optimal number of components $m_i^*(i=1, \ldots, n)$ is a nondecreasing function of P.

PROOF:

(a) By hypothesis,

†It has been seen that y_i was an increasing function of y_1. Therefore, if two solutions exist for the system of Equations (1), all the components of one of these solutions are larger than the corresponding components in the other solution.

††This comes from the fact that:

$$p_j^{M_j} - p_j^{M_j+m} = (p_j^{m_j} - p_j^{M_j+1}) + (p_j^{M_j+1} - p_j^{M_j+2}) + \ldots + (p_j^{M_j+m-1} - p_j^{M_j+m})$$

$$= p_j^{M_j}(1-p_j)(1+p_j+\ldots+p_j^{m-1}) \leqq mp_j^{M_j}(1-p_j).$$

(i) $$PR - C \geqslant PR' - C' \qquad \text{or } P(R - R') \geqslant C - C'$$

and

(ii) $$P'R' - C' \geqslant P'R - C' \qquad \text{or } P'(R - R') \leqslant C - C'.$$

Combining (i) and (ii), we get:

$$P'(R-R') \leqslant P(R-\bar{R}') \qquad \text{or } R \leqslant R' \qquad \text{to prove our assertion.}$$

(b) Let us suppose that for some j, $m_j'^* = m_j^* - m$ (m a positive integer). We know that $R' \geqslant R$, from part (a), so that with $m_j'^* < m_j^*$, then $R_j' > R_j$.

By hypothesis,

(iii) $$PR_j(1-p_j^{m_j'^*}) > PR_j(1-p_j^{m_j^*}) + mc_j$$

or

$$PR_j(p^{m_j^*} - p^{m_j}) > mc_j.$$

Similarly,

(iv) $$P'R_j'(1-p_j^{m_j'^*}) > P'R_j'(1-p_j^{m_j^*}) - mc_j$$

or

$$P'R_j'(p_j^{m_j'^*} - p_j^{m_j^*}) < mc_j.$$

Grouping (iii) and (iv), we get

$$PR_j(p_j^{m_j'^*} - p_j^{m_j^*}) > P'R_j'(p_j^{m_j'^*} - p_j^{m_j^*}),$$

which leads to $m'^*_j > m_j^*$ and thus to a contradiction.

3. ROUNDING THE NONINTEGER SOLUTION

In [1], Fan, Wang, Tillman, and Huang use a discrete local maximum principle to get the following system of equations:

$$P\prod_{i=1}^{n}(1-y_i) = a_i(1-y_i)/y_i \qquad (i=1, \ldots, n) \text{ with } a_i = c_i/\log(p_i) \text{ and } y_i = p_i^{m_i}(0 \leqslant y_i \leqslant 1).$$

They then use the Falsi iteration method to solve these Equations [1]. The noninteger m_i's are easy to determine from the y_i's and this noninteger solution is then rounded to the nearest integer solution.

First, it should be noted that the use of control theory in this application is absolutely redundant. Theorem I shows that the same system of equations can be obtained directly by setting the first derivatives of the objective function equal to zero. This is naturally far simpler than the process described in [1].

The number of operations necessary to get the noninteger solution by the Falsi iteration method is quite small. It is therefore quite understandable that the computation time was not affected by the size of the problems. With the data listed in the appendix, problems from 3 to 99 stages were all solved in a time of about 1 second, compilation time excluded (about 3 seconds).*

However, the solutions were frequently far from optimal. For the data listed in Appendix I† and with $P = 40{,}000$, we got the following results:

TABLE 1

Percentage Difference between Exact and Approximate Solutions

Number of stages	Approximative profit	Optimal profit	Difference (%)
25	26374	26413	0.15
34	19054	19081	0.15
35	18081	18484	2.2
38	16514	17453	5.6
42	14472	16078	11.
49	6568	8122	22.
57	756	2111	179.
59	− 120	1128	∞
62	− 934	400	∞

Table 1 shows a gap increasing with n, the number of stages, between the approximate solution obtained from rounding the noninteger optimal solution to the nearest integer solution (i.e., 2.49 is rounded to 2 and 3.51 to 4, for example) and the integer optimal solution. The main cause of error was due (for more than 34 stages) to component 7 (which is also component 33 and 36–42). For n greater than 34, the noninteger solution for this problem is about 1.49 which is then rounded to 1. But the optimal solution requires two of this component at the appropriate stages. This table therefore shows that the process can sometimes lead to significant errors especially as n gets large. Moreover, if we compute for each stage a maximum number of components M_i, by simply applying Theorem V, it has been found that, for all the problems considered, M_i was a better solution than that given by the rounding process. However, although we found this rounding process dangerous, it is too widely used for many other reliability problems (See [2] and [7] for example) to be discarded completely.

Theorem III shows that the system of Equations (1) admits at most two solutions, while Theorem IV shows that only one local maximum can exist and only in the region $\sum_{i=1}^{n} p_i^{m_i} \leq 1$. If, for such a point, the objective function is positive, this local maximum is also a global maximum. Otherwise the solution

*The computer used was a Univac 1108. The times given in this paper do not include the compilation time (3 seconds for the first method, up to 6 seconds for the adapted branch and bound and Ketelle's algorithms.) All the problems have been solved with the set of data listed in the appendix I. Only the data for the first 62 stages are listed in this appendix.

†The data were chosen at random for the first 30 stages and then to accentuate difficulties which arise when rounding the noninteger solution as proposed by Fan, *et al.*

to the problem is the trivial solution $\{0\}$. From Theorem II, we see that the function is concave around the local maximum. From these facts, we can deduce that if the system of Equations (1) admits two solutions, only the largest must be computed. Therefore, we can suppose that the Falsi method used by Fan, *et al.*, always give the largest root of (1); i.e., the possible solution. If the objective function is positive at that point, the point is a maximum (because the objective function admits at least a maximum if it can have positive values). If it is negative, we stop and take $\{0\}$ as the solution. From the concavity of the objective function around the local maximum, we deduce that the real solution usually lies among the corners of the unit hypercube containing that point; i.e., if $\{h_i\}$ $(i=1, \ldots, n)$ represents the noninteger solution of the problem, the optimal solution usually lies among all the corners of the hypercube $\{[h_i], [h_i]+1\}$.†

Therefore, the computation of the noninteger optimum allows us to only consider 2^n possible solutions. Moreover, if $[h_i] \geqslant M_i$ for some i, we know that $[h_i]+1$ will never be in the optimal solution and therefore, the number of possible solutions can be decreased to 2^k with $k \leqslant n$, if there are $(n-k)$ stages for which $[h_i] \geqslant M_i$. This very important property will be used later for speeding computations with the two other methods investigated in this paper and will lead ultimately to an algorithm which gives the exact solution to the problem with no increase in computation time whatever the number of stages may be.

4. KETELLE'S ALGORITHM

In [5], Ketelle described an algorithm for optimizing multistage reliability systems based on the concept of an "undominated" or optimal sequence. He first constructs a sequence of undominated allocations for the first two stages; i.e., all the possible combinations of cost and reliability (C, R) such that there does not exist another combination with better reliability and smaller cost. These terms constitute the optimal sequence for the first two stages and they are ordered with increasing cost. He then adds the third stage and, using the former optimal sequence, creates the optimal sequence for the first three stages. He continues adding stages until the optimal sequence for the n stages is obtained. (See an example in Appendix II.)

It is obvious that the optimal solution to our problem corresponds to one term of the final optimal sequence for the n stages. Therefore, we applied Ketelle's algorithm to compute the optimal sequence for all n stages. Then the objective function $(PR-C)$ was computed for each of the terms of the n stage optimal sequence. The term which had the largest value of the objective function yielded the optimal solution.

The computation time for this process however, increased exponentially with the number of stages. The number of terms in the optimal sequence increases very quickly with the number of stages†† and the number of computations increases faster than $\sum_{i=1}^{n} t_i$ where t_i is the number of terms in the optimal sequence on the first i stages. At the limit, this could lead to computation time doubling each time two stages would be added.

†By $[x]$, we mean the largest integer number contained in x. In all the examples, the optimum solution was found to be one corner of the unit hypercube.

††The increase was sometimes as much as 50 percent from the optimal sequence on i stages to the optimal sequence on $(i+1)$ stages.

Moreover, all the indices $\{m_i\}$ corresponding to each term of the optimal sequence had to be kept in memory. The limits of core space available in the computer were rapidly reached so that it was not possible to solve problems with more than about 20 stages using this process (for $P = 40{,}000$). For such small problems, the computation time was still small (about 4 seconds).

Larger problems were considered without computing all the indices corresponding to each term of the optimal sequence; i.e., we computed the maximum value of the objective function but could not retrieve the optimal sequence which yielded this optimum. For 31, 32, and 33 stages, the times were 14, 20, and 30 seconds, respectively; thus showing the exponential increase mentioned formerly. Fortunately, we can apply the following theorem which significantly decreases the number of terms in the otpimal sequence at each stage to speed the computations and decrease the storage needed for solving the problem.

THEOREM VII: Given an optimal sequence on the i first stages with $q_i(j)$ and $C_i(j)$ being the reliability and cost, respectively, of the jth term of this optimal sequence, then if for

$$j < k, \; PQ_i(j) - C_i(j) > PQ_i(k) - C_i(k),$$

the kth term of this partial optimal sequence will never be part of the optimal solution.

PROOF: Let us suppose that the optimal solution is made with the kth term of the optimal sequence on the i first stages. Let Q^i and C^i be the corresponding reliability and cost on the remaining $(n-1)$ stages in the optimal solution.

The objective function at the optimal solution is therefore: $PQ_i(k)Q^i - C_i(k) - C^i$. We then show that, if we replace in this optimal solution, the i first m_i's by those corresponding to the jth term of the optimal sequence on the i first stages, we get a better solution and hence a contradiction:

By hypothesis, $PQ_i(j)$ is larger than $PQ_i(k) - C_i(k)$, or $P[Q_i(k) - Q_i(j)] < C_i(k) - C_i(j)$. Therefore,

$$P[Q_i(k) - Q_i(j)]Q^i < C_i(k) - C_i(j) = [C_i(k) + C^i] - [C_i(j) + C^i],$$

or

$$PQ_i(j)Q^i - C_i(j) - C^i > PQ_i(k)Q^i - C_i(k) - C^i,$$

which proves the theorem.

Thus, after each stage i, we could eliminate from the optimal sequence all the terms k for which there exists a lower term, j, in the sequence $(j < k)$ for which

$$PQ_i(j) - C_i(j) > PQ_i(k) - C_i(k).$$

With this process, the number of terms in the dominating sequences were not always increasing with the number of stages. Computation time did improve and larger problems could be solved. However, for large problems, core limitations remained the main obstacle and it seems likely that tapes would have to be used. The following table gives the computation times in seconds and the number of terms of the final optimal sequence for problems with various values of P ranging from 20,000 up to 80,000.

The number of terms in the optimal sequence is not given for all the values of P and n considered in this table. It can be noticed that, other things remaining equal, the larger P, the larger the number of terms in the optimal sequence. This is a direct consequence of Theorem VI. As P increases, R increases and the bounds M_i on the number of components are nondecreasing functions of P. Moreover, less terms are eliminated from the optimal sequence of each stage.*

The following table also shows the maximum size of the problems solved by this method for the different values of P. (For $P = 20{,}000$ larger problems could have been solved, but the objective function was not far from 0 with 35 stages.)

TABLE 2

Computation Time and (in parenthesis) the Number of Terms of the Optimal Sequence

Number of stages	Values of P (000 omitted)			
	20	40	60	80
20			4	4(334)
21			4(313)	5(353)
22			4(320)	6(361)
23			5(361)	7(418)
24		5	6(360)	7(423)
25		5	7(366)	8(442)
26	6	6	8(420)	
27	6	7	9(422)	
28	6	8	10(428)	
29	7	9	13(428)	
30	8(132)	12	13(443)	
31	8(119)	14(369)	15(482)	
32	8(111)	15		
33	8(93)	17		
34	8(69)	19(403)		
35	8(39)			

In other words, this table shows that the larger P, the smaller the size of the problems we can solve without using tapes. However, the computation time is small and this method is useful for small problems ($n < 20$).

A final modification based on the last remark of the former section produced an algorithm, which, using Ketelle's optimal sequence, was able to solve fairly large problems in very short time. The noninteger solution was computed first. Then the optimal sequence is constructed at each stage with only the two integers lying around the noninteger solution at each stage. Moreover, terms were eliminated from the optimal sequence by the use of the former theorem. This permitted relatively small optimal sequences.†

*$PQ_i(k) - C_i(k) \leqq PQ_i(j) - C_i(j)\ (j < k)$ does not necessarily lead to $P'Q_i(k) - C_i(k) \leqq P'Q_i(j) - C_i(j)$ for $P' > P$, but does it if $P' < P$.

†We did not compute M_i, the maximum number of components in this procedure for if M_i coincides with the smallest of the two integers about the noninteger solution, terms of the optimal sequence involving the largest of the two integers will disappear rapidly from the optimal sequence in the next stages. Thus a self elimination process will take place.

Table 3 gives the results (time in seconds, and number of terms in the optimal sequence) of this process for $P=40{,}000$. Again terms have been eliminated from the optimal sequence by the use of Theorem VII.

TABLE 3

Number of stages	Time	Number of terms in the optimal sequence
35	2	
41	3	
45	4	167
48	4	159
49	4	186
50	4	183
51	4	182

Problems with $n>51$ could not be solved because of core limitations.

5. BRANCH AND BOUND ALGORITHM (See a brief description in Appendix III).

Fundamentally, there are no special difficulties in applying the branch and bound algorithm described in [4] to this particular problem. First the maximum number of components at each stage $\{M_i\}$ is determined by use of Theorem V. The initial solution and corresponding upperbound are $\{M_i\}$ and $P\left\{\prod_{i=1}^{n}(1-p_i^{M_i})\right\}-\sum_{i=1}^{n}c_iM_i$, respectively (or $\{0\}$ if the former solution gives a negative objective function). When we go forward, from stage j to stage $(j+1)$ the number of components at stage $(j+1)$ is set equal to 1. If $(j+1)=n$ however, we must find the number of components which actually maximizes the current solution. Therefore, the maximal number of components at the last stage is very important for the computation time. The minimum cost on the remaining stages, when we are at stage j, is $\sum_{i=j+1}^{n} c_i$. The backtracking remains identical to that described in [4]. The computations for this problem are more complicated than in the basic algorithm since PQ_j-C_j must be computed at each stage, prohibiting the use of logarithms for speeding the computations. Moreover, as we do not know the desired reliability, at each forward step, we always have to take 1 as the minimum number of components for the new stage. Therefore, the number of possible solutions increases very rapidly with the number of stages.

Table 4 shows the results for $P=40{,}000$:

TABLE 4

No. of stages	20	21	22	23	24	25	26
Time..............	4	5	7	12	19	46	242

These results are very much influenced by the maximum number of components at the last stage. (For 26 stages, this number is 7). However, for large problems which have $\{0\}$ as the optimal solution, the

time can become very short, because the backtrackings are done immediately and the number of possible solutions to consider is small.

But, for this branch and bound procedure, the use of the noninteger solution can again speed up the process by limiting the number of possible solutions. However, computation time still increases exponentially with the number of stages. But, there is never a core problem with this branch and bound algorithm. Table 5 shows the computational results of this last method with the aid of a noninteger solution, for $P = 40{,}000$.

TABLE 5

No. of Stages	30	31	32	33	34	35	36	37	38	39	40	41	42
Computation Time..............	5	8	9	10	16	18	24	36	48	66	96	155	212

It is not clear that, other things remaining constant, computation time increases with P. For a given number of stages, time depends mainly on the number of possible solutions, $2^k (k \leq n)$. This is shown in Table 6 which, for 31 stages, shows the different times corresponding to different values of P.

TABLE 6

P (000 omitted)	38	36	34	32	30	28	26	24	22	20
Time..............	5	5	8	5	8	23	83	98	67	104

From the above table, we can deduce that there were more possible solutions when $P = 20{,}000$ than when $P = 40{,}000$.

The only advantage so far of the branch and bound algorithm with respect to Ketelle's algorithm, is that, for a given number of stages, time does not increase monotonically with P, as it does with Ketelle's method. Moreover, if a solution is known for a given value of P, Theorem VI shows that the optimal solution for any larger value P' is always larger than or equal to the optimal solution for P. Therefore, continuous increases in P are easy to solve with this algorithm.

It is possible to further decrease the number of possible solutions by the use of the following results. Let us suppose that we have been able to determine a minimal number of components at each stage 1, $L_i (i = 1, \ldots, n)$. Let us call $R_{i,L}$ the minimal reliability on the system made of all the stages except stage i; i.e.,

$$R_{i,L} = \left[\prod_{j=1}^{n} (1 - p_j^{L_j}) \right] / (1 - p_i^{L_i}).$$

The bound on the minimal number of components at each stage can eventually be improved by:

THEOREM VIII: If $[p_i^{L_i} - p_i^{L_i+1}]$ is larger than $c_i / (PR_{i,L})$, the minimum number of components at stage i can be increased from L_i to $(L_i + 1)$.

Similarly, if a maximum number of components M_i has been determined for each stage $i (i = 1, \ldots, n)$ and $R_{i,M}$ represents the maximum reliability on the whole system, stage i expected, we have:

THEOREM IX: If $p_i^{M_i-1} - p_i^{M_i}$ is smaller than $c_i/(PR_{i,M})$, the maximum number of components at stage i can be decreased from M_i to $(M_i - 1)$.

PROOFS: The proofs of Theorems VIII and IX follow directly from the proof of Theorem V.

Using Theorems VIII and IX, we learn that:

(1) If the minimum solution* is $\{1, \ldots, 1\}$, and for problems with less than 30 stages ($P=40{,}000$), the optimal solution was gotten directly by the repeated use of Theorem VIII. For $n > 30$, Theorem VIII led to no improvement in the minimum solution and therefore was useless. In the first case, problems were solved in less than 1 second.

(2) If the noninteger solution $\{h_i\}$ is computed, a useful minimum branch to consider becomes $\{[h_i]\}$ $(i=1, \ldots, n)$. Applying Theorem VIII to this initial minimum solution, we solved all attempted problems within 3 seconds and the first feasible solution generated always turned out to be the optimum solution. Problems up to 77 variables have been solved with this algorithm. Different values of P were tried and in every case, the algorithm led to the optimal solution in 3 seconds.† If problems of larger size were submitted, it would also be possible to use Theorem IX for insuring that both minimum and maximum solutions become equal and thus that no other solution than the optimal one must be tried by the branch and bound technique.

Consequently, although we cannot prove that this algorithm will always work as fast as it did in our numerous experiments and that it will always lead to the optimal solution, we consider it the only way of solving large problems without any difficulty and in a very short time.

6. ACKNOWLEDGMENTS

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*By minimum solution we mean the minimum branch of the tree of possible solutions, i.e., the solution made of the smallest possible values at each stage which need to be considered in the branching process.

†The method remains heuristic as the optimal solution does not necessarily lie among the corner of the unit hypercube.

Appendix I

Data of the Problems

Stage number	Reliability $1-p_i$	Cost c_i	Stage number	Reliability $1-p_i$	Cost c_i
1	0.86	200	32	0.75	173
2	0.95	196	33	0.99	170
3	0.84	194	34	0.80	152
4	0.92	189	35	0.84	149
5	0.82	182	36	0.99	170
6	0.75	173	37	0.99	170
7	0.99	170	38	0.99	170
8	0.80	152	39	0.99	170
9	0.84	149	40	0.99	170
10	0.79	147	41	0.99	170
11	0.96	143	42	0.99	170
12	0.85	142	43	0.75	643
13	0.92	140	44	0.79	847
14	0.81	136	45	0.80	152
15	0.98	132	46	0.84	149
16	0.85	126	47	0.96	143
17	0.68	125	48	0.85	142
18	0.87	124	49	0.92	140
19	0.89	122	50	0.56	140
20	0.92	135	51	0.87	150
21	0.91	139	52	0.89	150
22	0.99	140	53	0.73	173
23	0.86	129	54	0.95	40
24	0.95	140	55	0.73	143
25	0.73	143	56	0.56	145
26	0.86	145	57	0.73	210
27	0.87	150	58	0.93	141
28	0.89	150	59	0.81	101
29	0.73	143	60	0.99	100
30	0.54	197	61	0.59	23
31	0.64	134	62	0.85	100

Appendix II

In this appendix, we give an example of the construction of an undominated sequence for a four-stage problem.

Data: $(p_1, p_2, p_3, p_4) = (0.01, 0.02, 0.01, 0.05)$ and $(c_1, c_2, c_3, c_4) = (10, 10, 5, 5)$: $P = 100{,}000$. In what follows, we give the optimal sequence, with the cost of each term of this sequence and the approximate unreliability of these terms.

(A) Use Theorem V to compute the maximum number of components at each stage. This gives: $M = (2, 3, 2, 4)$. The undominated sequences that we will compute, will not include more components at each stage than the above maxima.

(B) Optimal sequence on the two first stages:

Term Number	1	2	3	4
Composition	(1, 1)	(1, 2)	(2, 2)	(2, 3)
Cost	20	30	40	50
Unreliability	0.03	0.01	0.0005	0.0001

(C) Optimal sequence on the three first stages:

1	2	3	4	5	6	7
(1, 1, 1)	(1, 1, 2)	(1, 2, 1)	(1, 2, 2)	(1, 2, 3)	(2, 2, 2)	(2, 3, 2)
25	30	35	40	45	50	60
0.04	0.03	0.02	0.01	0.01	0.0006	0.0002

(D) Optimal sequence on the four stages:

1	2	3	4	5	6	7	8	9	10	11
(1, 1, 1, 1)	(1, 1, 1, 2)	(1, 1, 2, 2)	(1, 2, 1, 2)	(1, 2, 2, 2)	(1, 2, 2, 3)	(2, 2, 2, 2)	(2, 2, 2, 3)	(2, 2, 2, 4)	(2, 3, 2, 3)	(2, 3, 2, 4)
30	35	40	45	50	55	60	65	70	75	80
0.09	0.043	0.033	0.023	0.013	0.01	0.0031	0.0007	0.0006	0.0003	0.0002

This example shows that the number of terms of the undominated sequence may increase very quickly with the number of stages.

Appendix III

Branch and Bound Algorithm

(A) Compute a minimum and maximum number of components at each stage, L_i and M_i, respectively $(i = 1, \ldots, n)$.

(B) The algorithm generates branches among the tree of possible solutions. We call pseudosolution a solution for which all the components have not yet been determined. Enter the algorithm at stage 1 with the pseudosolution: $PS_1 = \{L_1\}$.

(C) Description of a current iteration. Suppose that we are currently at stage j, i.e., that the current pseudosolution we are now considering is made of j numbers $(m_1, \ldots, m_j)$. Let us suppose that the current best solution found so far is $(\bar{m}_1, \ldots, \bar{m}_n)$ corresponding to an objective function $\overline{0}$.

Test step:

(a) If $j < n$, see if there exists a possibility for a solution better than $\overline{0}$ with the branch actually under consideration. If not go to the backtracking step. If yes, go to the forward step.

(b) If $j = n$, compare the actual solution with $\overline{0}$; update $\overline{0}$ and $(\bar{m}_1, \ldots, \bar{m}_n)$, if the new solution is better and go to the backtracking step.

Backtracking step:

(a) If $j = 1$, terminate.

(b) Otherwise do $j = j - 1$. If $m_j = M_j$ go to the backtracking step again. Otherwise do $m_j = m_j + 1$ and go to the test step.

Forward step:

(a) If $j < n - 1$, do $j = j + 1$ and $m_j = L_j$ go to the test step.

(b) If $j = n -$, do $j = n$ and take m_j in such a way as to maximize the objective function with the $(n-1)(m_1, \ldots, m_{n-1})$ now under consideration.

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TOLERANCE INTERVALS FOR UNIVARIATE DISTRIBUTIONS

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ABSTRACT

A review of univariate tolerance intervals is presented from an application-oriented point of view. Both β-content and β-expectation intervals are defined and considered. Standard problems are discussed for the distribution-free case and with various distributional assumptions (normal, gamma, Poisson) which occur most frequently in practice. The determination of sample size is emphasized. A number of examples are used to illustrate the types of problems which permit solutions with the excellent tables now available.

1. INTRODUCTION

Everyone who has been exposed to a formal statistics course is acquainted with the concept of a confidence interval. In those courses we learn that such intervals are constructed from observed values of a random sample in an attempt to capture an unknown parameter of a distribution. A tolerance interval is also an interval computed from observed values of random sample. Now, however, the objective is to capture a given fraction of the distribution from which the sample is drawn. With the confidence interval we draw an inference about a parameter of a distribution and with a tolerance interval we draw an inference about a proportion of a distribution.

In order to be more specific let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution having a density function $f(x; \theta)$ and distribution function $F(x; \theta)$, where θ is a parameter of one or more dimensions. Further, let $U_1 = U_1(X_1, X_2, \ldots, X_n)$, $U_2 = U_2(X_1, X_2, \ldots, X_n)$, where $U_1 < U_2$, and $W = F(U_2; \theta) - F(U_1; \theta)$. Also, let β and γ be given positive numbers less than 1 (but usually close to 1). If U_1, U_2 are determined so that

$$Pr(W \geqslant \beta) = Pr\left[\int_{U_1}^{U_2} f(x; \theta)\, dx \geqslant \beta\right] \geqslant \gamma, \tag{1.1}$$

where $Pr(W \geqslant \beta)$ does not depend on θ, then (u_1, u_2) is called a β-content tolerance interval at level γ. (Small letters are used to denote observed values. Thus, the observed random sample is $x_1, x_2, \ldots, x_n$, u_1 and u_2 are the observed values of U_1 and U_2, and w is the observed value of W.) On the other hand, if U_1, U_2 are determined so that

$$E(W) = E\left[\int_{U_1}^{U_2} f(x; \theta)\, dx\right] \geqslant \beta, \tag{1.2}$$

where $E(W)$ does not depend on θ, then (u_1, u_2) is called a β-expectation interval. When $Pr(W \geqslant \beta)$

and $E(W)$ do not depend on the density $f(x; \theta)$, then (u_1, u_2) is called a distribution-free tolerance interval. The numbers u_1, u_2 are usually referred to as lower and upper tolerance limits, respectively. If $U_1 = -\infty$ (or the smallest value which X can assume, say, a) so that the interval becomes (a, u_2) or if $U_2 = \infty$ (or the largest value which X can assume, say b) yielding (u_1, b), then the term one-sided tolerance interval is used while (u_1, u_2) is called a two-sided tolerance interval. When X has a discrete distribution the integrals in (1.1) and (1.2) are replaced by sums and the tolerance limits u_1, u_2 are included in the interval.

It is perhaps of interest to mention some specific problems involving tolerance intervals. A manufacturer of $\frac{1}{2}$-inch bolts may wish to determine two numbers u_1 and u_2, such that he can state with a high degree of confidence (say, $\gamma = 0.95$) that (u_1, u_2) contain at least $\beta = 0.90$ of the distribution of diameters. Alternatively he may decide to determine the interval in a manner such that on the average intervals so calculated will contain at least $\beta = 0.90$ of the distribution. A maker of flashlight batteries may desire a number u_1 constructed so that in the long run, such intervals will contain at least 0.90 (β) of the distribution of life lengths at least 0.95 (γ) of the time. A cherry cannery which attempts to maintain a net weight (including fruit and juice) of 16 ounces per can may wish an interval (u_1, u_2) for the drained weight of the fruit. Perhaps the interval is to be determined so that in the long run such intervals will contain 90 percent or more $(\beta = 0.90)$ of the drained weights at least 95 percent $(\gamma = 0.95)$ of the time. Finally, consider a problem involving a manufacturer of automobiles. Suppose that his vehicles which are shipped to dealers have X major defects (perhaps those defects which require corrective work by the dealer) per unit. The manufacturer would like to have a number u_2 based upon a random sample of n automobiles determined so that with a high degree of confidence (γ) intervals $(0, u_2)$ contain the number of major defects for at least 0.90 (β) of his automobiles.

There are, of course, various ways in which U_1, U_2 can be constructed so that condition (1.1) or (1.2) is satisfied. It is natural to prefer intervals which are "best" in some sense. One criterion of best β-content tolerance intervals was introduced by Goodman and Madansky [5]. The consequence of their definition is that w is "as close as possible" to β. Intervals possessing this property are called "most stable." For one-sided tolerance intervals Guenther [8] has shown that the tolerance limit which yields the most stable interval can be related to and derived from the best test of an equivalent hypothesis testing problem. In this review we omit the definitions and theory of the above mentioned papers. The tolerance limits which we will use in the specific cases are generally based upon good statistical procedures.

Let us return to the bolt example for a moment. Recall that $\beta = 0.90$, $\gamma = 0.95$, and U_1, U_2 were to be determined so that

$$Pr(W \geq 0.90) \geq 0.95.$$

Perhaps it is intuitively obvious that the smaller the sample size n, the larger we expect the interval (u_1, u_2) to be. Now suppose that the manufacturer not only desires to have the above condition satisfied, but also wishes that (u_1, u_2) not be "too long" since long intervals tend to make his manufacturing process seem inferior. One way to enforce this second condition is to require that the probability be low that the tolerance interval captures a "high" proportion of the distribution. To be specific let us suppose that the bolt manufacturer also desires to have

$$Pr(W \geq 0.99) \leq 0.05.$$

To generalize the above discussion we must first specify four constants $\beta_0 < \beta_1, \gamma_0, \gamma_1$. Then we require that

(1.3) $$Pr(W \geq \beta_0) \geq \gamma_0$$

and

(1.4) $$Pr(W \geq \beta_1) \leq \gamma_1.$$

As we shall demonstrate in various cases, (1.3) and (1.4) determine a minimum sample size n.

Sample size problems can also be considered in conjunction with β-expectation intervals. Suppose that the bolt manufacturer wishes to determine (u_1, u_2) so that $E(W)=0.90$. He may also require that with high probability W will be close to $E(W)$. For example, he could insist that

$$Pr(0.90-0.10 \leq W \leq 0.90+0.05) \geq 0.99$$

or more generally, given d_1, d_2, δ that

(1.5) $$Pr[E(W)-d_1 \leq W \leq E(W)+d_2] \geq \delta.$$

Condition (1.5) in conjunction with (1.2) will determine a minimum sample size. It may be that only a lower bound on W is required so that (1.5) is replaced by

(1.6) $$Pr[W \geq E(W) - d_1] \geq \delta.$$

On the other hand only an upper bound may be required so that (1.5) is replaced by

(1.7) $$Pr[W \leq E(W) + d_2] \geq \delta.$$

In section 2 we will consider some tolerance interval problems in the distribution-free case. In later sections it will be assumed that X is governed by a specific distribution depending upon one or more unknown parameters.

2. DISTRIBUTION-FREE TOLERANCE INTERVALS

We first assume that X has a continuous distribution, but that the form of the density is unknown. Let $X_{(1)}$ be the smallest of the random sample $X_1, X_2, \ldots, X_n$, $X_{(2)}$ be the second smallest, $\ldots$, $X_{(n)}$ be the largest. After observing $x_1, x_2, \ldots, x_n$ these numbers can be arranged from smallest to largest yielding $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$. Now suppose we take $u_1 = x_{(i)}$ for the case in which we seek an interval of the type (u_1, ∞). For a given β, γ we wish to determine the relationship which n and i must satisfy so that condition (1.1) will hold. Let $x_{1-\beta}$ be the point of the distribution of X which is exceeded with probability β. Obviously w will be greater than or equal to β if $x_{(i)}$ is less than or equal to $x_{1-\beta}$. In order for this to happen we need one of the following:

$$x_{(i)} \leq x_{1-\beta} \quad \text{and } x_{(i+1)} > x_{1-\beta}$$

$$x_{(i+1)} \leqslant x_{1-\beta} \quad \text{and } x_{(i+2)} > x_{1-\beta}$$
$$\vdots$$
$$x_{(n-1)} \leqslant x_{1-\beta} \quad \text{and } x_{(n)} > x_{1-\beta}$$
$$x_{(n)} \leqslant x_{1-\beta}.$$

In other words i or more of the independently chosen x's must be less than or equal to $x_{1-\beta}$. That is, $P(W \geqslant \beta)$ = probability of i or more successes in a binomial type experiment with probability of success $1-\beta$. Hence, we can write

$$Pr(W \geqslant \beta) = \sum_{k=i}^{n} \binom{n}{k} (1-\beta)^k \beta^{n-k} = E(i;\, n,\, 1-\beta) \tag{2.1}$$

and (1.1) becomes

$$E(i;\, n,\, 1-\beta) \geqslant \gamma. \tag{2.2}$$

Inequality (2.2) can be solved for n if i is given or for i if n is given. Such solutions are obtained by inspection in one of the good binomial tables (Harvard [14], Ordnance Corps [23]). According to our definition $(x_{(i)}, \infty)$ is a β-content tolerance interval at level γ and since the solution of (2.2) does not depend on knowledge of the density of X, the interval is distribution-free.

If the two conditions (1.3), (1.4) are imposed, then we would solve

$$E(i;\, n,\, 1-\beta_0) \geqslant \gamma_0 \tag{2.3}$$

and

$$E(i;\, n,\, 1-\beta_1) \leqslant \gamma_1 \tag{2.4}$$

for minimum n and corresponding i. This solution is also obtained by inspection in binomial tables.

A similar type argument can be used to show that $(-\infty, x_{(j)})$ is a distribution-free β-content tolerance interval at level γ and (2.2), (2.3), and (2.4) are replaced by

$$E(n-j+1;\, n,\, 1-\beta) \geqslant \gamma, \tag{2.5}$$

$$E(n-j+1;\, n,\, 1-\beta_0) \geqslant \gamma_0, \text{ and} \tag{2.6}$$

$$E(n-j+1;\, n,\, 1-\beta_1) \leqslant \gamma_1. \tag{2.7}$$

Also, it can be shown that $(x_{(i)}, x_{(j)})$ is a two-sided distribution-free β-content tolerance interval where (2.2), (2.3), (2.4) are replaced by

(2.8) $$E(n-j+i+1;\ n,\ 1-\beta) \geqslant \gamma,$$

(2.9) $$E(n-j+i+1;\ n,\ 1-\beta_0) \geqslant \gamma_0, \text{ and}$$

(2.10) $$E(n-j+i+1;\ n,\ 1-\beta_1) \leqslant \gamma_1.$$

We observe that (2.8), (2.9), and (2.10) also hold for both one-sided cases if we agree to set $i=0$ when seeking only an upper tolerance limit, $j=n+1$ when seeking only a lower tolerance limit.

It is fairly easy to verify that (2.2), (2.5), (2.8) and (2.3), (2.6), (2.9) are still correct even though X has a discrete distribution. However, (2.4), (2.7), (2.10) are no longer applicable so that we are unable to solve sample size problems depending on these relationships.

In summary, the three main problems associated with distribution-free β-content tolerance intervals are: (1) Given i, or j, or both and β, γ find the minimum n which satisfies (2.2), or (2.5), or (2.8). (2) Given n, β, γ, find i, or j, or $j-i$ for which (2.2), or (2.5), or (2.8) is satisfied. (3) Given β_0, β_1, γ_0, γ_1 find the minimum n and accompanying i, or j, or $j-i$ which satisfies both conditions (2.3) and (2.4), or (2.6) and (2.7), or (2.9) and (2.10). Frequently all three types of problems can be solved by inspection using a good binomial table.

EXAMPLE 2.1: If $\beta=0.90$, $\gamma=0.95$, $n=50$ find the distribution-free one-sided and two-sided β-content tolerance intervals with $Pr(W \geqslant 0.90) \geqslant 0.95$ being as nearly satisfied as possible. If $\beta=0.90$, $\gamma=0.95$, $i=1$, $j=n-1$, find the distribution-free one-sided and two-sided β-content tolerance intervals with minimum sample size.

SOLUTION: For the first problem conditions (2.2), (2.5), and (2.8) are $E(i;\ 50,\ 0.10) \geqslant 0.95$, $E(51-j;\ 50,\ 0.10) \geqslant 0.95$, $E(51-j+i;\ 50,\ 0.10) \geqslant 0.95$. By inspection in a binomial table we find $i=2$, $51-j=2$ or $j=49$, and $51-(j-i)=2$ or $j-i=49$. Then for the one-sided cases the desired intervals are $(x_{(2)}, \infty)$, $(-\infty, x_{(49)})$. For the two-sided case we have $(x_{(1)}, x_{(50)})$.

For the second problem conditions (2.2), (2.5), and (2.8) are $E(1;\ n,\ 0.10) \geqslant 0.95$, $E(2;\ n,\ 0.10) \geqslant 0.95$, $E(3;\ n,\ 0.10) \geqslant 0.95$. By observation we find from the Ordnance Corps [23] table that to satisfy the inequalities we must have $n \geqslant 29$, $n \geqslant 46$, $n \geqslant 61$, respectively. Then the desired intervals are $(x_{(1)}, \infty)$ obtained with $n=29$, $(-\infty, x_{(45)})$ obtained with $n=46$, and $(x_{(1)}, x_{(60)})$ obtained with $n=61$.

EXAMPLE 2.2: If $\beta_0=0.90$, $\gamma_0=0.95$, $\beta_1=0.99$, $\gamma_1=0.05$, find the distribution-free one-sided and two-sided β_0-content tolerance intervals with minimum n.

SOLUTION: Consider the two-sided case. Conditions (2.9) and (2.10) are $E(n-j+i+1;\ n,\ 0.10) \geqslant 0.95$, $E(n-j+i+1;\ n,\ 0.01) \leqslant 0.05$. The solution is obtained by trial starting with $n-j+i+1=1$ and increasing this quantity a unit (or more) at a time until a solution is found. With $n-j+i+1=1$ we find by observation with a binomial table that we must have $n \geqslant 22$ and $n \leqslant 5$ to satisfy the two inequalities so that no solution is possible. With $n-j+i+1=2$, we need $n \geqslant 38$ and $n \leqslant 35$, again an impossibility. With $n-j+i+1=3$, we need $n \geqslant 52$ and $n \leqslant 82$. The desired n is 52 and $j-i=50$. The tolerance interval can be $(x_{(2)}, x_{(52)})$ or $(x_{(1)}, x_{(51)})$.

With only a lower tolerance limit we need $i=3$, $n=52$, the interval being $(x_{(3)}, \infty)$. With only an upper tolerance limit we need $n-j+1=3$, $n=52$, $j=50$ so that the interval is $(-\infty, x_{(50)})$.

We next consider distribution-free β-expectation tolerance intervals. Results from the β-content case suggest that a two-sided interval should be of the form $(x_{(i)}, x_{(j)})$. Using such an interval it can be shown that $E(W)=(j-i)/(n+1)$. Consequently (1.2) becomes

$$\frac{j-i}{n+1} \geqslant \beta. \tag{2.11}$$

If we let $i=r$, $j=n-s+1$, then (2.11) can be written as

$$n \geqslant \frac{r+s}{1-\beta} - 1. \tag{2.12}$$

To incorporate condition (1.5) into the problem write $E(W)=(j-i)/(n+1)=1-(r+s)/(n+1)$. Then the left hand side of that inequality can be written as

$$Pr\left(W \geqslant 1-\frac{r+s}{n+1}-d_1\right) - Pr\left(W \geqslant 1-\frac{r+s}{n+1}+d_2\right).$$

Using (2.1) on the last two probabilities converts (1.5) to

$$E\left(r+s;\, n,\, \frac{r+s}{n+1}+d_1\right) - E\left(r+s;\, n,\, \frac{r+s}{n+1}-d_2\right) \geqslant \delta. \tag{2.13}$$

(If $p=(r+s)/(n+1)-d_2 \leqslant 0$, then the second binomial sum in (2.13) is 0.) Given $r+s$, d_1, d_2, δ, (2.13) can be solved by trial for minimum n using binomial tables. Formulas (2.12) and (2.13) also apply to one-sided intervals $(x_{(i)}, \infty)$, $(-\infty, x_{(j)})$ if we set $s=0$ in the first case, $r=0$ in the second.

If only a lower bound on W is required (1.6) becomes

$$E\left(r+s;\, n,\, \frac{r+s}{n+1}+d_1\right) \geqslant \delta \tag{2.14}$$

and if only an upper bound is required (1.7) becomes

$$E\left(r+s;\, n,\, \frac{r+s}{n+1}-d_2\right) \leqslant 1-\delta. \tag{2.15}$$

Again these formulas apply to the one-sided cases if either s or r is set equal to zero.

EXAMPLE 2.3: Using $i=1$, $j=n$, $(r=s=1)$ and $\beta=0.80$, find a two-sided β-expectation tolerance interval with minimum n. Find the minimum n if we also require that (1.5) be satisfied with $d_1=d_2=0.10$, $\delta=0.90$.

SOLUTION: Using (2.12) we get $n \geqslant (2/0.20)-1=9$. Hence $n=9$ and the interval is $(x_{(1)}, x_{(9)})$. If in addition n must satisfy (1.5), then according to (2.13) we need

$$E\left(2;\, n,\, \frac{2}{n+1}+0.10\right) - E\left(2;\, n,\, \frac{2}{n+1}-0.10\right) \geqslant 0.90.$$

With $n=9$ we find $E(2;\, 9,\, 0.30)-E(2;\, 9,\, 0.10)=0.80400-0.61258=0.18142$. With $n=19$ we get $E(2;\, 19,\, 0.20)-E(2;\, 19,\, 0)=0.91713$. With $n=18$ we have $E(2;\, 18, 0.2053)-E(2;\, 18, 0.0053)=0.9089-$

$0.0047 = 0.9042$ where the first value was obtained by interpolation in a binomial table, the second by the Poisson approximation. With $n=17$ we get $E(2;\ 17, 0.2111) - E(2;\ 17, 0.0111) = 0.9013 - 0.0175 = 0.8838$. Hence we need $n=18$ to satisfy both conditions and the interval is $(x_{(1)}, x_{(18)})$.

3. TOLERANCE INTERVALS FOR THE NORMAL DISTRIBUTION

Let X be a normal random variable with unknown mean μ and unknown variance σ^2. (The less interesting cases in which one parameter is known have been discussed by Guenther [6].) Tolerance limits of the type $\bar{x} \pm ks$, where $\bar{x}$ and s are the usual unbiased estimates of the mean and variance and k is a constant, have been used since first suggested by Wilks [28]. Such limits are an intuitively reasonable choice and can be justified mathematically for the one-sided β-content problem.

For the case in which only a lower tolerance limit is sought we will use $(\bar{x} - ks, \infty)$. We can write

$$W = 1 - F(\bar{X} - kS;\ \mu, \sigma^2) = \int_{(\bar{X}-\mu-kS)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz.$$

Since $(\bar{X} - \mu)/\sigma$ has a standard normal distribution, $W \geqslant \beta$ implies $(\bar{X} - \mu - kS)/\sigma < z_{1-\beta} = -z_\beta$. (If Z has a standard normal distribution, then z_p is defined by $Pr(Z < z_p) = p$.) Consequently, for the β-content problem (1.1) becomes

(3.1) $$Pr\left(\frac{\bar{X} - \mu - kS}{\sigma} < -z_\beta\right) = \gamma.$$

(Equality can be achieved with a continuous random variable.) It is easy to verify that (3.1) can be rewritten as

(3.2) $$Pr(T < k\sqrt{n}) = \gamma,$$

where

$$T = \frac{\dfrac{\sqrt{n}(\bar{X} - \mu)}{\sigma} + \sqrt{n}\, z_\beta}{S/\sigma}.$$

The random variable T has a noncentral t-distribution with $n-1$ degrees of freedom and noncentrality parameter $\sqrt{n} z_\beta$. Owen [20, Table 2] has prepared a very extensive table giving k to three decimal places. Entries include all combinations of

$$\gamma = 0.995, 0.99, 0.975, 0.95, 0.90, 0.75, 0.50, 0.25, 0.10, 0.05, 0.025, 0.01, 0.005$$

$$P = \beta = 0.75, 0.90, 0.95, 0.975, 0.99, 0.999, 0.9999, 0.99999$$

$$n = 2(1)200(5)400(25)1000, 1{,}500, 2{,}000, 3{,}000, 5{,}000, 10{,}000, \infty.$$

With this table most of the problems concerning one-sided β-content tolerance intervals are readily solved. If n is given together with γ and β, then k is read directly if β is one of the P values given above. If β is not in that list, then k must be found. Linear interpolation on z_P usually provides sufficient accuracy. (The Owen publication includes a description of a more accurate, but much more complicated alternative.)

If we impose conditions (1.3) and (1.4), then we would solve

$$Pr(T < k\sqrt{n}) = \gamma_0 \qquad \text{if } \beta = \beta_0 \tag{3.3}$$

and

$$Pr(T < k\sqrt{n}) \leqslant \gamma_1 \qquad \text{if } \beta = \beta_1 \tag{3.4}$$

for minimum n and accompanying k. (Due to the fact that n is an integer, equality cannot be achieved in both (3.3) and (3.4). We will adopt the convention of requiring equality in the former.) If both β_0, β_1 are among the listed entries for P in Owen's table, then n and k are found by observation. The minimum sample size is obtained by increasing n until the k which produces equality in (3.4) is just greater than the k which produces equality in (3.3). If one or both of the β's are missing from the list, then the k's can be found by linear interpolation as mentioned above.

A good approximate formula for the minimum sample size was obtained by Wallis [25]. The discussion leading to (3.1) enables us to write for (1.3) and (1.4)

$$Pr(\bar{X} - \mu - kS < -\sigma z_{\beta_0}) = \gamma_0, \; Pr(\bar{X} - \mu - kS < -\sigma z_{\beta_1}) \leqslant \gamma_1.$$

Then, assuming that $\bar{X} - kS$ is approximately normally distributed with mean $\mu - k\sigma$ and variance $(\sigma^2/n)(1 + k^2/2)$, one quickly finds that

$$n \geqslant \left(1 + \frac{k^2}{2}\right)\left[\frac{z_{\gamma_0} - z_{\gamma_1}}{z_{\beta_1} - z_{\beta_0}}\right]^2, \; k = \frac{z_{\gamma_0} z_{\beta_1} - z_{\gamma_1} z_{\beta_0}}{z_{\gamma_0} - z_{\gamma_1}}. \tag{3.5}$$

Such an approximation is useful if at least one of β_0, β_1 is missing from the P entries used by Owen since considerable interpolation may be eliminated.

Similar analysis for the upper tolerance limit case leads to the interval $(-\infty, \bar{x} + ks)$ where k and n are determined by the procedures described above.

EXAMPLE 3.1: If $\beta = 0.90$, $\gamma = 0.95$, $n = 10$ find an upper one-sided β-content tolerance limit. If $\beta_0 = 0.90$, $\beta_1 = 0.99$, $\gamma_0 = 0.95$, $\gamma_1 = 0.05$ find an upper one-sided tolerance limit with minimum n.

SOLUTION: To solve the first problem we enter Owen's [20] Table 2 with $\gamma = 0.95$, $\beta = P = 0.90$, $n = 10$. We read $k = 2.355$ so that the desired limit is $\bar{x} + 2.355s$.

For the second problem we solve (3.3) and (3.4) for minimum n and corresponding k. By observation we find from Owen's table that if $n = 26$

$$Pr(T < 1.824\sqrt{26}) = 0.95 \qquad \text{if } P = \beta_0 = 0.90$$

$$Pr(T < 1.809\sqrt{26}) = 0.05 \qquad \text{if } P = \beta_1 = 0.99$$

while if $n = 27$

$$Pr(T < 1.811\sqrt{27}) = 0.95 \qquad \text{if } P = \beta_0 = 0.90$$

$$Pr(T < 1.817\sqrt{27}) = 0.05 \qquad \text{if } P = \beta_1 = 0.99.$$

Thus the required solution is $n=27$, $k=1.811$ and the tolerance limit is $\bar{x}+1.811s$.

EXAMPLE 3.2: If $\beta_0=0.85$, $\beta_1=0.96$, $\gamma_0=0.90$, $\gamma_1=0.05$ find the minimum n and corresponding k for a lower one-sided tolerance limit. (These figures were used by Faulkenberry and Weeks [3].)

SOLUTION: Since neither $\beta_0=0.85$ nor $\beta_1=0.96$ is included among the P's of Owen's [20] Table 2, we use linear interpolation on z_P to obtain the k's. To eliminate some of the trial we first estimate n from (3.5). We get

$$k=\frac{(1.282)(1.751)-(-1.645)(1.037)}{1.282-(-1.645)}=1.350$$

and

$$n \geqslant \left(1+\frac{1.350^2}{2}\right)\left[\frac{1.282+1.645}{1.751-1.037}\right]^2=32.1\ .$$

Let us first try $n=33$. Interpolating for k on z_P yields

$$Pr(T<1.360\ \sqrt{33})=0.90 \qquad \text{with } P=\beta_0=0.85$$

$$Pr(T<1.359\ \sqrt{33})=0.05 \qquad \text{with } P=\beta_0=0.96$$

while with $n=34$ we get

$$Pr(T<1.355\ \sqrt{34})=0.90 \qquad \text{with } P=\beta_0=0.85$$

$$Pr(T<1.364\ \sqrt{34})=0.05 \qquad \text{with } P=\beta_0=0.96.$$

(For both $n=33$ and $n=34$ the first interpolation was based upon k values for $P=0.75$, 0.90 while the second was based upon k values for $P=0.95$, 0.975.) Hence the solution is $n=34$, $k=1.355$ and the tolerance limit is $\bar{x}-1.355s$. (Apparently Faulkenberry and Weeks did not use the Owen table and found $n=35$, $k=1.348$.)

We next consider two-sided β-content tolerance intervals of the type $(\bar{x}-ks,\ \bar{x}+ks)$. No simple probability statement in terms of a tabulated distribution (corresponding to (3.2) in the one-sided case) can be obtained. Two approaches have been taken. First, using a high speed computer exact values of k satisfying $Pr(W\geqslant\beta)=\gamma$ can be found (and have been for some β, γ, $n=2(1)10$ by Webb and Friedman [26]). Second, several approximations for k have been devised (the best known, being due to Wald and Wolfowitz [24]) and some tables have been prepared using these approximations. (For such tables see Bowker [2], Owen [19], Weissberg and Beatty [27].) One of the simplest approximations for k is due to Howe [15]. He has shown that a reasonably accurate value is

$$k = \lambda u, \tag{3.6}$$

where

$$\lambda = z_{(1+\beta)/2}\left(1+\frac{1}{n}\right)^{1/2}, \quad u = \left[\frac{n-1}{\chi^2_{n-1;\,1-\gamma}}\right]^{1/2}.$$

(If Y_ν has a chi-square distribution with ν degrees of freedom, then $Pr(Y_\nu < \chi^2_{\nu;\,p}) = p$.) Although λ and u can be calculated from standard tables (i.e., Harter's [13] table), Owen [20, Table 3.5] has given the former quantity to three decimal places for all combinations of

$$P = (1+\beta)/2 = 0.75,\ 0.90,\ 0.95,\ 0.975,\ 0.99,\ 0.999,\ 0.9999,\ 0.99999,$$

$$n = 1(1)200(5)400(25)1{,}000,\ 1{,}500,\ 2{,}000,\ 3{,}000,\ 5{,}000,\ 10{,}000,\ \infty,$$

and the latter (Owen [19], pp. 134–137) to four decimal places for all combinations of

$$\gamma = 0.50,\ 0.75,\ 0.90,\ 0.95,\ 0.99,\ 0.999,$$

$$f = n-1 = 1(1)150(2)250(5)500(10)800(20)1{,}000(1{,}000)10{,}000.\ \infty.$$

Howe has also shown that the approximation is improved by replacing the k of (3.6) by

$$k_1 = k\left[1+\frac{n-3-\chi^2_{n-1;\,1-\gamma}}{2(n+1)^2}\right]^{1/2}. \tag{3.7}$$

In fact, even for $n = 2$ (3.7) gives very nearly the exact value of k given by Webb and Friedmann [26]. Consequently for practical purposes it is not necessary to prepare exact tabulations of k.

If the two conditions (1.3) and (1.4) are imposed, the minimum sample size and corresponding k can be found by trial. To do this increase n until the k which produces equality in (1.4) is no smaller than the k which produces equality in (1.3). Using the k given by (3.6) for both cases leads to

$$\frac{\chi^2_{n-1;\,1-\gamma_1}}{\chi^2_{n-1;\,1-\gamma_0}} \leqslant \left[\frac{z_{(1+\beta_1)/2}}{z_{(1+\beta_0)/2}}\right]^2 \tag{3.8}$$

an equality which n must satisfy. If the improved approximation (3.7) is used, then (3.8) is replaced by

$$\frac{\chi^2_{n-1;\,1-\gamma_1}}{\chi^2_{n-1;\,1-\gamma_0}}\left[\frac{2(n+1)^2+n-3-\chi^2_{n-1;\,1-\gamma_0}}{2(n+1)^2+n-3-\chi^2_{n-1;\,1-\gamma_1}}\right] \leqslant \left[\frac{z_{(1+\beta_1)/2}}{z_{(1+\beta_0)/2}}\right]^2. \tag{3.9}$$

The minimum n's satisfying (3.8) and (3.9) are not apt to differ by much since, except for very small n, the quantity in the bracket on the left hand side of (3.9) will be nearly 1. Probably one would first solve (3.8) and then, if necessary, increase n until (3.9) is satisfied (as demonstrated in the following example).

EXAMPLE 3.3: If $\beta_0=0.80$, $\gamma_0=0.90$, $\beta_1=0.95$, $\gamma_1=0.05$, find the minimum n and accompanying k for a two-sided β_0-content tolerance interval.

SOLUTION: First we find the smallest n satisfying (3.8) which for this problem is

$$\frac{\chi^2_{n-1;\,0.95}}{\chi^2_{n-1;\,0.10}} \leqslant \left[\frac{z_{0.975}}{z_{0.90}}\right]^2 = \left[\frac{1.960}{1.282}\right]^2 = (1.529)^2 = 2.338.$$

For $n=24$ the left side of the inequality yields $35.17/14.85=2.368>2.338$ while for $n=25$ it is $36.42/15.66=2.326<2.338$. Hence based upon (3.8) the minimum sample size is 25. If we use (3.9) with $n=25$, the left hand side yields $2.362>2.338$. With $n=26$ we get $2.320<2.338$. Based upon the improved approximation the minimum sample size is 26.

Next we find k from (3.6). We get $\lambda=1.306$, $u=1.232$ from which $k=1.609$. Using the improved approximation we get $k_1=1.609(1.0045)^{1/2}=1.613$. Hence the desired interval is $(\bar{x}-1.613s, \bar{x}+1.613s)$.

We next consider β-expectation tolerance intervals. As in the β-content case we will use tolerance limits of the form $\bar{x}\pm ks$, where k is chosen to satisfy (1.2) (with equality). For the two-sided case it can be shown that we can write

$$E(W)=Pr(\bar{X}-kS<X<\bar{X}+kS). \tag{3.10}$$

Now with a little algebra the latter expression becomes

$$E(W)=Pr\left[-k\left(1+\frac{1}{n}\right)^{-1/2}<T<k\left(1+\frac{1}{n}\right)^{-1/2}\right],$$

where

$$T=\frac{(X-\bar{X})/\sigma(1+1/n)^{1/2}}{S/\sigma}$$

has a t-distribution with $n-1$ degrees of freedom. To make $E(W)=\beta$ requires that

$$k=t_{n-1;\,(1+\beta)/2}\left(1+\frac{1}{n}\right)^{1/2}. \tag{3.11}$$

(If T has a t-distribution with ν degrees of freedom, then $Pr(T<t_{\nu;p})=p$.) For the one-sided case the result is

$$k=t_{n-1;\,\beta}\left(1+\frac{1}{n}\right)^{1/2}. \tag{3.12}$$

Wilks [28] gave the result (3.11), but without derivation. (Perhaps he felt none was necessary since the equivalence of the two sides of (3.10) is easily established by writing an expression for each.) An extensive table of (3.12) was published by Owen [20, Table 3.4] giving k to three decimal places for the same $\beta=P$ and n listed following (3.6).

Let us turn our attention to a few possibilities for determining sample size with β-expectation intervals. First consider one-sided intervals of the type $(-\infty, \bar{x}+ks)$. Then, with $E(W)=\beta$, (1.6) becomes

$$Pr(W \geqslant \beta - d_1) \geqslant \delta \tag{3.13}$$

which is condition (1.1) encountered in the β-content case with β replaced by $\beta - d_1$, γ replaced by δ. Hence, according to (3.2) we need

$$Pr(T < k\sqrt{n}) \geqslant \delta, \tag{3.14}$$

where T has a noncentral t-distribution with $n-1$ degrees of freedom and noncentrality parameter $\sqrt{n}z_{\beta-d_1}$. The two conditions (3.12) and (3.14) will determine a minimum n and accompanying k. If β is an entry in Owen's [20] Table 3.4 and $\beta - d_1$, δ are entries in his Table 2, then the problem is quickly solved by observation. The sample size is increased until the k from the former table just exceeds the k which produces equality in (3.14). If either β or $\beta - d_1$ are not entries in the above mentioned tables, we can construct our own tables (by linear interpolation on z_P in Table 2). If condition (1.7) is used, (3.14) is replaced by

$$Pr(T < k\sqrt{n}) \leqslant 1 - \delta, \tag{3.15}$$

and the noncentrality parameter is $\sqrt{n}z_{\beta+d_2}$. Now n is increased until the k from Owen's Table 3.4 is just less than the k which yields equality in (3.15). If condition (1.5) is incorporated into the problem, exact solutions with existing tables are more difficult. We can, however, use the Wallis [25] approximation used to derive (3.5). Thus we have that

$$Pr(W \geqslant \beta) \cong Pr\left[Z > \frac{(z_\beta - k)\sqrt{n}}{(1+k^2/2)^{1/2}}\right],$$

and (1.5) becomes

$$Pr\left[\frac{\sqrt{n}\,(z_{\beta-d_1} - k)}{(1+k^2/2)^{1/2}} < Z < \frac{\sqrt{n}\,(z_{\beta+d_2} - k)}{(1+k^2/2)^{1/2}}\right] \geqslant \delta \tag{3.16}$$

(where again Z is the standard normal random variable). Then, using k's given by (3.12), increase n until (3.16) is satisfied. For intervals of the type $(\bar{x} - ks, \infty)$ all of the comments and formulas of this paragraph apply without modification.

When the tolerance interval is two-sided and condition (1.6) or (1.7) is imposed, the solution for minimum n is fairly easy. By use of Howe's result (3.6) gives immediately

$$\left[\frac{z_{(1+\beta-d_1)/2}}{t_{n-1;\,(1+\beta)/2}}\right]^2 \leqslant \frac{\chi^2_{n-1;\,1-\delta}}{n-1} \tag{3.17}$$

for (1.6), and

$$\left[\frac{z_{(1+\beta+d_2)/2}}{t_{n-1;\,(1+\beta)/2}}\right]^2 \geqslant \frac{\chi^2_{n-1;\,\delta}}{n-1} \tag{3.18}$$

for (1.7). After the minimum n is determined by trial, the k of (3.11) is found in the usual way. To incorporate condition (1.5) with two-sided intervals we use formulas (2.6) and (2.7) from the Howe [15] paper. If in the latter of those two formulas we approximate by replacing $\bar{x}^2$ by $1/n$ and neglect terms

of the order $1/n^2$ (essentially what Wald and Wolfowitz [24] did) we get

$$Pr\left[S^2 \geqslant \left(1+\frac{1}{n}\right)^{1/2} \frac{z^2_{(1+\beta)/2}}{k^2}\right] \cong Pr(W \geqslant \beta). \tag{3.19}$$

Then using (3.19) and the k of (3.11) we can replace (1.5) by

$$Pr\left[(n-1)\frac{z^2_{(1+\beta-d_1)/2}}{t^2_{n-1;(1+\beta)/2}} < Y_{n-1} < (n-1)\frac{z^2_{(1+\beta+d_2)/2}}{t^2_{n-1;(1+\beta)/2}}\right] \geqslant \delta, \tag{3.20}$$

where Y_{n-1} has a chi-square distribution with $n-1$ degrees of freedom. Again the minimum n is determined by trial. (The tables of Khamis and Rudert [16] are well adapted to evaluating incomplete chi-square integrals.) The left hand side of (3.20) is essentially the same probability obtained by Albert and Johnson [1]. Their limits were, however, given in a slightly less convenient form.

EXAMPLE 3.4: Find the minimum n and accompanying k for a one-sided β-expectation tolerance interval satisfying (1.6) if $\beta=0.90$, $d_1=0.15$, $\delta=0.99$. Repeat using condition (1.7) with $d_2=0.05$.

SOLUTION: Since $\beta = P = 0.90$ is an entry in Owen's [20] Table 3.4 and $\beta - d_1 = P = 0.75$, $\delta=\gamma=0.99$ are entries in his Table 2, we find the solution by observation. With $n=23$, $k=1.350$ yields the desired expectation and equality is achieved in (3.14) with $k=1.355$. Thus $n=23$ is too small. With $n=24$, $k=1.347$ yields the desired expectation and equality is achieved in (3.14) with $k=1.336$. Hence with $n=24$ the one-sided intervals are $(\bar{x}-1.347s, \infty)$ and $(-\infty, \bar{x}+1.347s)$.

For the second part $\beta=0.90$, $\beta+d_2=0.95$, $1-\delta=0.01$ are entries in the Owen tables. With $n=83$, $k=1.300$ produces the desired expectation while $k=1.299$ yields equality in (3.15) so n is too small. With $n=84$, $k=1.300$ again produces the desired expectation while $k=1.301$ yields equality in (3.15). Hence with $n=84$ the one-sided intervals are $(\bar{x}-1.300s, \infty)$ and $(-\infty, \bar{x}+1.300s)$.

EXAMPLE 3.5: Find the minimum n and accompanying k for a two-sided β-expectation interval satisfying (1.7) if $\beta=0.90$, $d_2=0.05$, $\delta=0.99$.

SOLUTION: Inequality (3.18) is

$$\left[\frac{z_{0.975}}{t_{n-1;0.95}}\right]^2 \geqslant \frac{\chi^2_{n-1;0.99}}{n-1}.$$

With $n=86$ the left side is $(1.960/1.663)^2=1.390$ while the right side yields $118.24/85=1.391$. With $n=87$ the left side is again 1.390 while the right side is $119.41/86=1.388$ so $n=87$. (The chi-square values are from Harter's [13] table, the t-values from Owen's [19] Table 2.1.) From Owen's [20] Table 3.4 we find $k=1.672$ and the interval is $(\bar{x}-1.672s, \bar{x}+1.672s)$.

EXAMPLE 3.6: Find the minimum n and accompanying k for a two-sided β-expectation interval satisfying (1.5) if $\beta=0.90$, $d_1=0.10$, $d_2=0.05$, $\delta=0.99$.

SOLUTION: Now n has to satisfy (3.20). The left side of that inequality is

$$Pr\left[(n-1)\frac{z^2_{.90}}{t^2_{n-1;.95}} < Y_{n-1} < (n-1)\frac{z^2_{.975}}{t^2_{n-1;.95}}\right].$$

With $n=89$ this reduces to $Pr(52.34 < Y_{88} < 122.3) = 0.9908 - 0.0009 = 0.9899$, where Y_{n-1} has a chi-square distribution with $n-1=88$ degrees of freedom (evaluation from Khamis-Rudert [16] Table). With $n=90$ we get $Pr(52.96 < Y_{89} < 123.8) = 0.9913 - 0.0009 = 0.9904$. Hence a sample size of 90 is required. Now $k=1.671$ and the interval is $(\bar{x}-1.671s, \bar{x}+1.671s)$.

4. TOLERANCE INTERVALS FOR THE GAMMA DISTRIBUTION

Let X be a random variable with the gamma density

$$f(x;\ \theta) = \frac{1}{\theta^{r/2}\Gamma(r/2)}\, x^{(r/2)-1}e^{-x/\theta},\ x>0,\ \theta>0,$$

where r is a known positive integer. (Actually we need only require $r>0$. For noninteger r our chi-square percentage points should be replaced by gamma percentage points, available from the table of Khamis and Rudert [16].) As an important special case, $r=2$ gives the exponential distribution which finds application in life testing.

It can be shown that there is sound mathematical justification for using tolerance limits of the form $u_1=k_1\bar{x}$, $u_2=k_2\bar{x}$ in the one-sided β-content cases (see Guenther [8]). We will also use the same type of limits in the other cases where, of course, the k's are chosen to fulfill specified conditions.

We consider first the one-sided β-content tolerance intervals. For the lower tolerance limit case we can write

$$W = 1 - F(k_1\bar{X};\theta) = \int_{2k_1n\bar{X}/n\theta}^{\infty} \frac{1}{2^{r/2}\Gamma(r/2)}\, (y_r)^{(r/2)-1}e^{-y_r/2}dy_r.$$

Since $Y_r = 2X/\theta$ has a chi-square distribution with r degrees of freedom (Y_ν denotes a chi-square random variable with ν degrees of freedom), $W \geqslant \beta$ implies that

$$\frac{k_1}{n}\frac{2n\bar{X}}{\theta} = \frac{k_1}{n} Y_{rn} < \chi^2_{r;\,1-\beta},$$

where Y_{rn} has a chi-square distribution with rn degrees of freedom. Hence (1.1) can be written

$$Pr\left(\frac{k_1}{n} Y_{rn} < \chi^2_{r;\,1-\beta}\right) = \gamma$$

or

$$Pr\left(Y_{rn} < \frac{n\chi^2_{r;\,1-\beta}}{k_1}\right) = \gamma. \tag{4.1}$$

Equation (4.1) is satisfied if $n\chi^2_{r;\,1-\beta}/k_1 = \chi^2_{rn;\,\gamma}$, from which

(4.2)
$$k_1 = \frac{n\chi^2_{r;\,1-\beta}}{\chi^2_{rn;\,\gamma}}\,.$$

If the two conditions (1.3) and (1.4) are imposed, then we need

$$Pr\left(Y_{rn} < \frac{n\chi^2_{r;\,1-\beta_0}}{k_1}\right) = \gamma_0,\ Pr\left(Y_{rn} < \frac{n\chi^2_{r;\,1-\beta_1}}{k_1}\right) \leqslant \gamma_1$$

from which

(4.3)
$$k_1 = \frac{n\chi^2_{r;\,1-\beta_0}}{\chi^2_{rn;\,\gamma_0}}\,,$$

and n must satisfy

(4.4)
$$\frac{\chi^2_{rn;\,\gamma_0}}{\chi^2_{rn;\,\gamma_1}} \leqslant \frac{\chi^2_{r;\,1-\beta_0}}{\chi^2_{r;\,1-\beta_1}}\,,$$

an inequality readily solved by trial. For the upper tolerance limit case a similar discussion yields

(4.5)
$$k_2 = \frac{n\chi^2_{r;\,\beta_0}}{\chi^2_{rn;\,1-\gamma_0}}\,,$$

and for the counterpart of (4.4) we get

(4.6)
$$\frac{\chi^2_{rn;\,1-\gamma_0}}{\chi^2_{rn;\,1-\gamma_1}} \geqslant \frac{\chi^2_{r;\,\beta_0}}{\chi^2_{r;\,\beta_1}}.$$

EXAMPLE 4.1: If $\beta_0=0.90$, $\gamma_0=0.95$, $\beta_1=0.99$, $\gamma_1=0.05$, and $r=5$ find the one-sided β_0-content tolerance intervals with minimum n.

SOLUTION: We consider the lower tolerance limit case first. Using (4.4) we must have

$$\frac{\chi^2_{5n;\,0.95}}{\chi^2_{5n;\,0.05}} \leqslant \frac{\chi^2_{5;\,0.10}}{\chi^2_{5;\,0.01}} = \frac{1.610}{0.5543} = 2.905.$$

With $n=3$ the left side of the inequality is $25.00/7.261=3.443$ while with $n=4$ it is $31.41/10.85=2.895$. Hence $n=4$ and $k_1=4\chi^2_{5;\,0.10}/\chi^2_{20;\,0.95}=4(1.610)/31.41=0.2050$. The interval is $(0.2050\bar{x}, \infty)$.

With an upper tolerance limit n must satisfy (4.6). Thus

$$\frac{\chi^2_{5n;\,0.05}}{\chi^2_{5n;\,0.95}} \geqslant \frac{\chi^2_{5;\,0.90}}{\chi^2_{5;\,0.99}} = \frac{9.236}{15.09} = 0.6121.$$

With $n=18$ the left side of the inequality is $69.13/113.1=0.6112$ while with $n=19$ it is $73.52/118.8=0.6189$. Hence $n=19$ and $k_2=19\chi^2_{5;\,0.90}/\chi^2_{95;\,0.05}=19(9.236)/73.52=2.387$. The interval is $(0, 2.387\bar{x})$.

We next consider two-sided β-content tolerance intervals of the type $(k_1\bar{x}, k_2\bar{x})$. If $w \geqslant \beta$ then

$$Pr(k_1\bar{x} < X < k_2\bar{x}) = Pr\left(\frac{k_1 y_{rn}}{n} < Y_r < \frac{k_2 y_{rn}}{n}\right) \geq \beta. \tag{4.7}$$

Suppose that k_1, k_2 are constants which permit (4.7) to be satisfied. Then (4.7) will hold for all $a \leq y_{rn} \leq b$ where a, b are the values of y_{rn} which yield equality. In order to satisfy (1.1) we must have

$$Pr(a < Y_{rn} < b) = \gamma, \tag{4.8}$$

and k_1, k_2 must be solutions of

$$Pr\left(\frac{ak_1}{n} < Y_r < \frac{ak_2}{n}\right) = Pr\left(\frac{bk_1}{n} < Y_r < \frac{bk_2}{n}\right) = \beta. \tag{4.9}$$

We can rewrite (4.9) as

$$Pr(K_1 < Y_r < K_2) = Pr(RK_1 < Y_r < RK_2) = \beta, \tag{4.10}$$

where $K_1 = bk_1/n$, $K_2 = bk_2/n$, $R = a/b$. As we shall demonstrate K_1 and K_2 can be found by trial with Khamis-Rudert [16] table. There are various ways to choose a and b. An easily obtained solution is

$$a = \chi^2_{rn;(1-\gamma)/2}, \; b = \chi^2_{rn;(1+\gamma)/2}. \tag{4.11}$$

Another choice is obtained by maximizing R subject to (4.8) thus requiring that a and b be "close" to each other so that the tolerance interval is "shortest." This leads to the condition

$$ag_{rn}(a) = bg_{rn}(b), \tag{4.12}$$

where $g_\nu(y)$ represents the chi-square density with ν degrees of freedom. Then to find a and b (4.8) and (4.12) are solved simultaneously. These solutions have been tabulated for some high values of γ (0.90, 0.95, 0.99, 0.999) by Lindley, East, and Hamilton [17], Pachares [21], and Tate and Klett [22]. It can be shown that (4.12) is equivalent to

$$Pr(a < Y_{rn+2} < b) = \gamma. \tag{4.13}$$

Then using (4.13) with (4.8), a and b can be found by trial from the Khamis-Rudert table (for an illustration see Guenther [7]). Except for small nr, the R obtained from (4.11) will differ very little from the one obtained by solving (4.8) and (4.13). Consequently in Example 4.2 we will use a and b given by (4.11).

If conditions (1.3) and (1.4) are imposed, then n can be found by trial. For each n which is selected find a, b from (4.11) with γ replaced by γ_0. Then with β replaced by β_0 solve (4.10) for K_1, K_2. In (4.9) let a', b' be the values of a, b obtained when β is replaced by β_1 (if such values exist). Further let $K_1' = b'k_1/n$, $K_2' = b'k_2/n$, $R' = a'/b'$ be the values of K_1, K_2, R which satisfy (4.10) when β is replaced by β_1. Then $K_2'/K_1' = k_2/k_1 = K_2/K_1 = c$ and (4.10) becomes

(4.14) $$Pr\ (K_1' < Y_r < cK_1') = Pr\ (R'K_1' < Y_r < R'K_2') = \beta_1,$$

which can be solved by trial for K_1', R'. From these we calculate $b' = nK_1'/k_1 = (K_1'/K_1)b$, $a' = R'b'$. Finally we compute

(4.15) $$Pr(a' < Y_{rn} < b')$$

increasing n until (4.15) is no larger than γ_1. After n has been determined we find $k_1 = nK_1/b$, $k_2 = nK_2/b$.

EXAMPLE 4.2: If $\beta_0 = 0.90$, $\gamma_0 = 0.99$, $\beta_1 = 0.95$, $\gamma_1 = 0.10$, and $r = 5$ find a two-sided β_0-content tolerance interval with minimum n based on (4.11).

SOLUTION: As indicated above we find the minimum n by trial. Suppose we start with $n = 10$. Then it can be verified that (4.15) yields about 0.66, considerably larger than $\gamma_1 = 0.10$ which we desire. Hence n must be increased. We will outline the calculations for $n = 13$, 14 which show that the minimum n is 14.

With $n = 13$ we first find from (4.11) that $a = \chi^2_{65;\,0.005} = 39.38$, $b = \chi^2_{65;\,0.995} = 98.11$. Then $R = 39.38/98.11 = 0.4014$ and (4.10) becomes

$$Pr(K_1 < Y_5 < K_2) = Pr(0.4014K_1 < Y_5 < 0.4014K_2) = 0.90,$$

which we solve for K_1, K_2 by trial with the Khamis-Rudert [16] table. Obviously we must have $K_1 \leqslant \chi^2_{5;\,0.10} = 1.610$ and we first try $K_1 = 1.60$, the nearest entry in the table. Then $RK_1 = 0.6422$, $Pr(Y_5 < 0.6422) = 0.01401$ and we need $Pr(Y_5 < RK_2) = 0.91401$. Linear interpolation yields $RK_2 = 9.644$ so that $K_2 = 24.03$. Finally we evaluate $Pr(1.60 < Y_5 < 24.03) = 0.99979 - 0.09875 = 0.90104$ which indicates that $K_1 > 1.60$. Trying $K_1 = 1.65$, the next entry in the table, gives $RK_1 = 0.6623$, $RK_2 = 9.676$, $K_2 = 24.11$, $Pr(1.65 < Y_5 < 24.11) = 0.99979 - 0.10486 = 0.89493$. Linear interpolation gives $K_1 = 1.609$, $K_2 = 24.04$. Next $c = 24.04/1.609 = 14.94$ and (4.14) becomes

$$Pr(K_1' < Y_5 < 14.94K_1') = Pr(R'K_1' < Y_5 < R'K_2') = 0.95$$

Now, we must have $K_1' \leqslant \chi^2_{5;\,0.05} = 1.145$. With $K_1' = 1.10$, $14.94K_1' = 16.43$ we find $Pr(1.10 < Y_5 < 16.43) = 0.94837$. With $K_1' = 1.05$, $14.94K_1' = 15.69$ we get $Pr(1.05 < Y_5 < 15.69) = 0.95064$. Linear interpolation yields $K_1' = 1.064$, $K_2' = 15.90$, but a further check yields $K_1' = 1.065$, $K_2' = 15.91$. Next we observe that $R'K_2' \geqslant \chi^2_{5;\,0.95} = 11.07$. If we try $R'K_2' = 11.1$, we find $R' = 0.6972$, $R'K_1' = 0.7425$, $Pr(0.7425 < Y_5 < 11.1) = 0.93112$. Hence $RK_2' = 11.1$ is too small. With further trial we find that with $R'K_2' = 13.2$ we get $R' = 0.8297$, $R'K_1' = 0.8836$, $Pr(0.8836 < Y_5 < 13.2) = 0.94982$ and with $R'K_2' = 13.3$ we get $R' = 0.8360$, $R'K_1' = 0.8903$, $Pr(0.8903 < Y_5 < 13.3) = 0.95020$. Linear interpolation gives $R'K_2' = 13.25$, $R' = 0.8328$, $R'K_1' = 0.8869$. Finally $b' = (1.065/1.609)98.11 = 64.94$, $a' = R'b' = 54.08$ and $Pr(54.08 < Y_{65} < 64.94) = 0.35216$. Since the latter probability is > 0.10, $n = 13$ is too small.

With $n = 14$ we use $a = \chi^2_{70;\,0.005} = 43.28$, $b = \chi^2_{70;\,0.995} = 104.2$, $R = 0.4154$ and find $K_1 = 1.608$, $K_2 = 23.31$, $c = 14.50$. In attempting to solve $Pr(K_1' < Y_5 < 14.50K_1') = 0.95$, no solution exists. After

a few trials it can be verified that $K_1'=0.987$ maximizes the probability at 0.94988. Hence (4.15) is 0 and the conditions of the problem are satisfied with $n=14$.

In conclusion $k_1 = nK_1/b = 14(1.608)/104.2 = 0.2160$, $k_2 = nK_2/b = 14(23.31)/104.2 = 3.132$ and the desired tolerance interval is $(0.2160\bar{x}, 3.132\bar{x})$.

Finally we consider β-expectation tolerance intervals. As in the β-content case we will use tolerance limits of the form $k_1\bar{x}$, $k_2\bar{x}$. For the two-sided case it can be shown that

$$E(W) = Pr(k_1\bar{X} < X < k_2\bar{X}) = Pr(k_1 < F_{r,\,rn} < k_2),$$

where

$$F_{r,\,rn} = \frac{2X/r\theta}{2n\bar{X}/rn\theta}$$

has an F-distribution with r and rn degrees of freedom. There are various ways to choose k_1 and k_2. Because of the format of available tables an easy choice yielding $E(W)=\beta$ is

$$k_1 = F_{r,\,rn;\,(1-\beta)/2},\quad k_2 = F_{r,\,rn;\,(1+\beta)/2}, \tag{4.16}$$

where the symbols in (4.16) are defined by

$$Pr(F_{r_1,\,r_2} < F_{r_1,\,r_2;\,p}) = p.$$

For the one-sided cases, which are probably of more practical value the choices are obviously

$$k_1 = F_{r,\,rn;\,1-\beta} \tag{4.17}$$

and

$$k_2 = F_{r,\,rn;\,\beta}. \tag{4.18}$$

For one-sided intervals it is routine to impose conditions (1.5), (1.6), and (1.7). Suppose, for example, we use (1.6) with the one-sided interval $(k_1\bar{x}, \infty)$. Then, we need $Pr(W \geqslant \beta - d_1) \geqslant \delta$ which is condition (1.1) encountered in the β-content case with β replaced by $\beta - d_1$, γ replaced by δ. Hence, to achieve equality in the probability statement we need according to (4.2)

$$k_1 = \frac{n\chi^2_{r;\,1-\beta+d_1}}{\chi^2_{rn;\,\delta}}. \tag{4.19}$$

To satisfy both conditions the k_1 of (4.19) has to be greater than or equal to the k_1 of (4.17). Thus, by trial we find the minimum n to satisfy

$$\frac{n\chi^2_{r;\,1-\beta+d_1}}{\chi^2_{rn;\,\delta}} \geqslant F_{r,\,rn;\,1-\beta}. \tag{4.20}$$

With condition (1.7) it can be verified that (4.20) should be replaced

$$\frac{n\chi^2_{r;\,1-\beta-d_2}}{\chi^2_{rn;\,1-\delta}} \leqslant F_{r,\,rn;\,1-\beta}. \tag{4.21}$$

Finally, if we want two bounds on W according to (1.5) then the minimum n is found by trial so that

$$Pr\left[\frac{n\chi^2_{r;\,1-\beta-d_2}}{F_{r,\,rn;\,1-\beta}} < Y_{rn} < \frac{n\chi^2_{r;\,1-\beta+d_1}}{F_{r,\,rn;\,1-\beta}}\right] \geqslant \delta, \tag{4.22}$$

an easy evaluation with the Khamis-Rudert table.

The results for intervals of the type $(0, k_2\bar{x})$ can be obtained from the above formulas after making the proper changes. In (4.19), (4.20), (4.21), and (4.22) replace $1-\beta$ by β. In the first three replace δ by $1-\delta$ and reverse the inequalities. In (4.22) interchange d_1 and d_2.

EXAMPLE 4.3: With $\beta=0.95$, $\delta=0.99$, $d_1=0.05$, $r=2$ find the β-expectation tolerance interval of the type $(k_1\bar{x}, \infty)$ which satisfies condition (1.6) with minimum n. Repeat using condition (1.7) with $d_2=0.025$. Finally, repeat the calculations once more using condition (1.5) with $d_1=0.05$, $d_2=0.025$.

SOLUTION: For the first problem we need according to (4.20),

$$\frac{n\chi^2_{2;\,0.10}}{\chi^2_{2n;\,0.99}} \geqslant F_{2,\,2n;\,0.05} = \frac{1}{F_{2n,\,2;\,0.95}}.$$

With $n=7$ we find

$$\frac{7\chi^2_{2;\,0.10}}{\chi^2_{14;\,0.99}} = \frac{7(0.2107)}{29.14} = 0.05061,\ F_{2,\,14;\,0.05} = \frac{1}{F_{14,\,2;\,0.95}} = \frac{1}{19.424} = 0.05148$$

(F value read from Owen's [19] Table 4.1, pp. 63–87) so n is too small. With $n=8$ the two numbers are, respectively, 0.05268 and 0.05146. Hence $n=8$, $k_1=F_{2,\,16;\,0.05}=0.05146$ and the interval is $(0.05146\bar{x}, \infty)$.

For the second part n must satisfy (4.21). That is, we must have

$$\frac{n\chi^2_{2;\,0.025}}{\chi^2_{2n;\,0.01}} \leqslant F_{2,\,2n;\,0.05}.$$

With $n=14$ we find

$$\frac{14\chi^2_{2;\,0.025}}{\chi^2_{28;\,0.01}} = \frac{14(0.05064)}{13.56} = 0.05228,\ F_{2,\,28;\,0.05} = \frac{1}{F_{28,\,2;\,0.95}} = \frac{1}{19.459} = 0.05139$$

and n is too small. With $n=15$ the two numbers are, respectively, 0.05081 and 0.05138. Hence $n=15$, $k_1=F_{2,\,30;\,0.05}=0.05138$ and the interval is $(0.05138\bar{x}, \infty)$.

Finally, for the third part n must satisfy (4.22). That inequality can be written

$$Pr[n\chi^2_{2;\,0.025}F_{2n,\,2;\,0.95} < Y_{2n} < n\chi^2_{2;0.10}\,F_{2n,2;0.95}] \geq 0.99.$$

Perhaps a good guess for n is the larger of the two results found in the first two parts of the problem. At any rate, with $n=14$ the left hand side of the inequality reduces to $Pr(13.80 < Y_{28} < 57.40)$. Entering the Khamis-Rudert table we find $Pr(Y_{28} < 57.40) = 0.99913$, $Pr(Y_{28} < 13.80) = 0.01145$. The desired probability is the difference $0.98768 < 0.99$, so n is too small. With $n=15$ we obtain $Pr(14.78 < Y_{30} < 61.51) = 0.99032 > 0.99$. Hence again $n=15$, $k_1=0.05138$ and the interval is $(0.05138\bar{x}, \infty)$.

In life testing problems it is often assumed that X, the length of life, has an exponential density (gamma with $r=2$) and that observations become available in order of magnitude. That is, one observes the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, where $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$. To reduce elapsed time in testing, experimentation is sometimes terminated after $m < n$ observations. Then tolerance intervals are based upon $\bar{X}'$ defined by $m\bar{X}' = \sum_{i=1}^{m} X_{(i)} + (n-m)X_{(m)}$ instead of $\bar{X}$. After replacing $rn = 2n$ degrees of freedom by $2m$, all calculations are as previously outlined.

5. TOLERANCE INTERVALS FOR DISCRETE DISTRIBUTIONS; POISSON CASE

In section 2 we have already mentioned that inequalities (2.2), (2.5), and (2.8) still hold when X has a discrete distribution. As in sections 3 and 4 with continuous distributions, we can do better if we are willing to assume that X has a discrete distribution of a given form. In general, mathematical results are much more difficult to obtain with discrete distributions, and as a consequence we will limit our discussion to β-content tolerance intervals. One-sided intervals were discussed by Zachs [29] for the Poisson, binomial, and negative binomial distributions.

In the introduction we mentioned an example involving an automobile manufacturer. He was interested in obtaining a β-content tolerance interval of the type $(0, u_2)$ for the distribution of the number of major defects (X) per vehicle. It is quite reasonable to assume that X has a Poisson distribution with unknown mean μ. Let $p(x;\mu)$ denote the Poisson probability function. Further, let $x_1, x_2, \ldots, x_n$ be the observed values of a random sample drawn from such a distribution and denote the sum by $y = \sum_{i=1}^{n} x_i$. It is, of course, well known that Y has a Poisson distribution with mean $n\mu$. Let y^* denote a specific observed value of y. Then, using Poisson tables (General Electric [4], Molina [18]) we can solve

(5.1)
$$\sum_{y=y^*}^{\infty} p(y;\, n\mu) = \gamma,$$

for μ obtaining, say, μ_2. It can be shown that the desired value of u_2 is the minimum value of u_2 which satisfies

(5.2)
$$\sum_{x=0}^{u_2} p(x;\,\mu_2) \geq \beta.$$

Because of the relationship between the Poisson and the the chi-square distribution, it follows from

(5.1) and (5.2) that the desired u_2 is the smallest value of u_2 which satisfies

(5.3) $$\chi^2_{2(u_2+1);\,1-\beta} \geqslant \frac{\chi^2_{2y^*;\,\gamma}}{n}.$$

To obtain a one-sided β-content tolerance interval of the type (u_1, ∞) for the Poisson distribution we solve

(5.4) $$\sum_{y=0}^{y^*} p(y;\, n\mu) = \gamma$$

for μ getting, say, μ_1. Then, the desired u_1 is the largest value of u_1 which satisfies

(5.5) $$\sum_{x=u_1}^{\infty} p(x;\, u_1) \geqslant \beta.$$

Alternately, the counterpart of (5.3) is

(5.6) $$\chi^2_{2u_1;\,\beta} \leqslant \frac{\chi^2_{2y^*+2;\,1-\gamma}}{n},$$

which is solved for the largest u_1 permitting a solution.

Two-sided intervals can be obtained with a slight modification of the above procedure. Now solve

(5.7) $$\sum_{y=y^*}^{\infty} p(y;\, n\mu) = \frac{1-\gamma}{2}, \quad \sum_{y=0}^{y^*} p(y;\, n\mu) = \frac{1-\gamma}{2}$$

for μ_1, μ_2. In terms of chi-square percentage points we have

(5.8) $$\mu_1 = \frac{\chi^2_{2y^*;\,(1-\gamma)/2}}{2n}, \quad \mu_2 = \frac{\chi^2_{2y^*+2;\,(1+\gamma)/2}}{2n}.$$

Then, any pair (u_1, u_2) which satisfy both

(5.9) $$\sum_{x=u_1}^{u_2} p(x;\, \mu_1) \geqslant \beta, \quad \sum_{x=u_1}^{u_2} p(x;\, \mu_2) \geqslant \beta$$

is a β-content tolerance interval at level γ. The solutions for (5.9) are obtained by observation from a Poisson table. Undoubtedly we would prefer to select a solution that makes $u_2 - u_1$ small as possible.

EXAMPLE 5.1: Suppose that in the automobile major-defect example that a random sample of 20 such automobiles yields the following number of defects: 4, 3, 3, 2, 10, 4, 3, 5, 8, 4, 3, 5, 3, 4, 8, 5, 5, 4, 4, 5. Find the distribution-free β-content tolerance interval of the type $(0, u_2)$ using $\beta = 0.90$, $\gamma = 0.80$. Then, find the appropriate interval under the assumption that the number of major defects follows a Poisson distribution.

Repeat the above finding a two-sided interval (u_1, u_2).

SOLUTION: In the distribution-free case according to (2.5) we need $E(21-j;\ 20,\ 0.10) \geqslant 0.80$. Using a binomial table we see that $21-j=1$, or $j=20$, is the smallest j to satisfy the inequality. Hence $u_2=x_{(20)}=10$ and (0, 10) is the desired interval.

To use the Poisson assumption we need $y^*=4+3+\ldots+5=92$. Then, (5.3) becomes

$$\chi^2_{2(u_2+1);\ 0.10} \geqslant \frac{\chi^2_{184;\ 0.20}}{20} = \frac{199.92}{20} = 9.996$$

(chi-square value from Harter's [13] table). With $u_2=7$ we find $\chi^2_{16;\ 0.10}=9.312$ and with $u_2=8$ we have $\chi^2_{18;\ 0.10}=10.86$. Hence, $u_2=8$ and the interval is (0, 8). (Alternatively, using (5.1) yields $n\mu_2=99.96$, $\mu_2=5.00$. Then (5.2) gives $u_2=8$ as before.)

For the two-sided distribution-free case according to (2.6) we need $E(21-j+i;\ 20,\ 0.10) \geqslant 0.80$. Now $j-i=20$, an impossibility. Hence $n=20$ is not a large enough sample size to yield a distribution-free solution.

Finally, we seek a two-sided interval under the Poisson assumption. From (5.8) we have

$$\mu_1 = \frac{\chi^2_{184;\ 0.10}}{40} = 3.997, \qquad \mu_2 = \frac{\chi^2_{186;\ 0.90}}{40} = 5.278.$$

Then, we find solutions for

$$\sum_{x=u_1}^{u_2} p(x;\ 3.997) \geqslant 0.90, \qquad \sum_{x=u_1}^{u_2} p(x;\ 5.278) \geqslant 0.90.$$

With the Molina [18] table it is easy to verify that the first inequality is satisfied for $u_1=0$, $u_2 \geqslant 7$, for $u_1=1$, $u_2 \geqslant 7$, for $u_1=2$, $u_2 \geqslant 9$, and the second inequality for $u_1=0$, $u_2 \geqslant 8$, for $u_1=1$, $u_2 \geqslant 8$, for $u_1=2$, $u_2 \geqslant 9$. Thus (0, 8), (1, 8), (2, 9) are all solutions of which one of the latter two would be preferred.

Although of less practical interest, β-content tolerance intervals for the binomial and negative binomial can be found in substantially the same way. For example, if X has a binomial distribution with parameters r, p, and probability function $b(x;\ r,\ p)$, then (5.1), (5.2), (5.4), (5.4) are replaced by

(5.10) $$\sum_{y=y^*}^{rn} b(y;\ rn,\ p) = \gamma,$$

(5.11) $$\sum_{x=0}^{u2} b(x;\ r,\ p_2) \geqslant \beta,$$

(5.12) $$\sum_{y=0}^{u2} b(y;\ rn,\ p) = \gamma, \text{ and}$$

(5.13) $$\sum_{x=u_1}^{r} b(x;\ r,\ p_1) \geqslant \beta.$$

If X has a negative binomial distribution with parameters c, p, and probability function $b^*(x; c, p)$, then in (5.10) to (5.13) replace b by b^*, r by c in the probability function. Also replace the upper limits in (5.10), (5.13) by ∞, the lower limit in (5.11) by c, and the lower limit in (5.12) by cn. For both distributions (c an integer) u_1 and u_2 are easily found with a good binomial table (i.e., the Ordnance Corps [23] table). (With large n (5.10) and (5.12) may be solved using various approximations for the binomial.) The necessary modifications for the two-sided case are obvious.

6. OTHER REVIEWS

The Guenther [6] report mentioned earlier contains a discussion similar to that given in sections 3 and 4 for the following distributions: (1) normal with unknown mean and known variance, (2) normal with known mean and unknown variance, (3) beta with one parameter being 1 and the other unknown, (4) Laplace with unknown dispersion parameter, (5) uniform on $(0, \theta)$ where θ is unknown, (6) exponential with unknown location parameter and known dispersion parameter, (7) exponential with both location and dispersion parameters unknown.

A series of four technical reports has been prepared by Guttman [9], [10], [11], [12]. These are written at a higher level of mathematical sophistication than this paper and there is considerably less emphasis on practical applications. Some results are given for multivariate problems (which we, of course, have not discussed). There appears to be very little duplication of the Guttman reports in this paper.

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PROPERTIES OF A MULTIFACILITY LOCATION PROBLEM INVOLVING EUCLIDIAN DISTANCES

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ABSTRACT

This paper considers a problem of locating new facilities in the plane with respect to existing facilities, the locations of which are known. The problem consists of finding locations of new facilities which will minimize a total cost function which consists of a sum of costs directly proportional to the Euclidian distances among the new facilities, and costs directly proportional to the Euclidian distances between new and existing facilities. It is established that the total cost function has a minimum; necessary conditions for a mimumum are obtained; necessary and sufficient conditions are obtained for the function to be strictly convex (it is always convex); when the problem is "well structured," it is established that for a minimum cost solution the locations of the new facilities will lie in the convex hull of the locations of the existing facilities. Also, a dual to the problem is obtained and interpreted; necessary and sufficient conditions for optimum solutions to the problem, and to its dual, are developed, as well as complementary slackness conditions. Many of the properties to be presented are motivated by, based on, and extend the results of Kuhn's study of the location problem known as the General Fermat Problem.

1. INTRODUCTION

All points in the plane, E_2, will be considered to be 2 by 1 vectors. Let the distinct, known points $Q_1, \ldots, Q_m$ represent the locations of the m existing facilities (EF's) and let the points $X_1, \ldots, X_n$ in E_2 represent the locations of the n new facilities (NF's). For any point Y in E_2, $|Y|$ denotes the Euclidian norm of Y, so that $|X_j - Q_i|$ represents the Euclidian distance between NFj and EFi, and $|X_j - X_k|$ represents the Euclidian distance between NFj and NFk. Costs directly proportional to the Euclidian distance between NFj and EFi are incurred, with w_{ji} being the nonnegative constant of proportionality, so that if the function $f_j(X_j)$ is defined for all X_j in E_2 by

$$f_j(X_j) = \sum_{i=1}^{m} w_{ji} |X_j - Q_i|, \qquad j = 1, \ldots, n,$$

then $f_j(X_j)$ may be considered to represent the total cost involving NFj and all EF's when NFj has the location X_j. Also, costs directly proportional to the Euclidian distance between NFj and NFk are incurred, with d_{jk} being the nonnegative constant of proportionality, so that if the function $f_0(X) = f_0(X_1, \ldots, X_n)$ is defined for all X_j in E_2 for $j = 1, \ldots, n$ by

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$$f_0(X) = \sum_{1 \leq j < k \leq n} d_{jk} |X_j - X_k|,$$

then $f_0(X_1, \ldots, X_n)$ may be considered to represent the total cost involving only NF's. The notation $X = (X_1, \ldots, X_n)$ will be understood to mean that X is the $2n$ by 1 column vector defined by $X^t = (X_1^t \ldots X_n^t)$ where the superscript t denotes the transpose operation.

With the above notation, the location problem may now be stated as follows: find a point $X^* = (X_1^*, \ldots, X_n^*)$ in E_{2n} to

$$\text{(1)} \qquad \text{minimize } f(X) \equiv f_0(X) + \sum_{j=1}^{n} f_j(X_j).$$

Subsequent reference to the function f will always be to the function f defined by (1).

The first version of the location problem (1) involving norms, in this case the l_1 norm, that is, involving rectilinear distances, appears to have been formulated by Francis [5], who later solved it for a special case [6]. Subsequently Cabot, Francis, and Stary [4] solved the general rectilinear distance version of (1) by converting it into two linear programming problems, each of which has the same structure; the dual of each of these linear programming problems is equivalent to a minimum cost network flow problem, which may be efficiently solved using Fulkerson's out-of-kilter algorithm. Independently, Wesolosky and Love [24] also observed that the rectilinear distance problem could be solved using linear programming. Pritsker and Ghare [18] have developed a direct search approach for solving the general rectilinear distance version of (1). When the points $Q_1, \ldots, Q_m$ are collinear, it can be shown that the problem of minimizing f reduces to one considered in [4]; thus for this case there are readily available solution techniques. White [25] has developed a method of solving the version of (1) involving the square of the Euclidian distances.

Vergin and Rogers [21], and Love [13], both give computational procedures for minimizing f. Each procedure is complicated by the fact that f is not everywhere differentiable; the Vergin and Rogers procedure sometimes gives suboptimum solutions. As will be seen, no differentiability problems occur with the dual of the location problem; thus it would appear that the development of a dual approach to solving the location problem might be of interest.

A number of the properties to be presented subsequently have been discovered independently by Wendell [22]; in almost all cases his proofs differ from the ones to follow. Wendell and Peterson [23] have made use of Peterson's [17] duality theory to show that the dual of the problem (1) can be obtained for an arbitrary norm defined on E_2, so that their duality is a generalization of the duality to be presented.

Note that in the problem (1) each NF may be located at any point in E_2. When the number of locations for new facilities is finite, and the possible locations are known, then (1) may be considered to be a special case of the quadratic assignment problem, for which there is a substantial facility location literature; see, for example, the work of Hillier and Connors [7], and of Nugent, et al. [16]. The problem (1) is distinguished from approaches to facility location involving the quadratic assignment problem by allowing each NF to be located at any point in E_2, and by using Euclidian distances.

Much of the analysis to follow is based on, motivated by, and extends Kuhn's analysis of the General Fermat Problem [10], [11], [12]; the problem may be stated as follows:

$$\text{minimize } F(X)=\sum_{i=1}^{m} w_i|X-P_i|,$$

where $P_i, \ldots, P_m$ are m distinct points in the plane, $w_1, \ldots, w_m$ are positive "weights," and X is also a point in the plane. It is convenient to state at this point several of Kuhn's results which will be used subsequently:

RESULT 1: Any point X^* which minimizes F is in the convex hull of the points $P_1, \ldots, P_m$.

RESULT 2: The point X^* minimizes F if and only if $R(X^*)=0$, where $R(X)$ is defined for all X in E_2 as follows:

$$R(X)=\sum_{i=1}^{m} (w_i/|X-P_i|)(P_i-X) \qquad \text{for } X\notin\{P_1, \ldots, P_m\}, ,$$

$$R(P_j)=[\max\ (|R_j|-w_j, 0)](R_j/|R_j|), \qquad j=1, \ldots, m,$$

where, by definition, for $j=1, \ldots, m$,

$$R_j=\sum_{i\neq j} (w_i/|P_i-P_j|)(P_i-P_j).$$

Subsequently the vector $R(X)$ will be referred to as Kuhn's modified negative gradient.

RESULT 3: The function F is strictly convex if, and only if, the points $P_1, \ldots, P_m$ are not collinear. When the points $P_1, \ldots, P_m$ are collinear, F is piecewise linear on the line through them, and strictly convex elsewhere.

Kuhn also develops a dual to the General Fermat Problem; rather than state it here, it will be pointed out as a special case of the dual of f to be developed subsequently. The dual to the General Fermat Problem has also been found independently by Witzgall and Rockafellar [12], [26], and appears implicitly in a dynamic programming procedure due to Bellman [1] for solving the problem.

For a further discussion of related literature on location problems, together with extensive bibliographies, see references [4], [12], and [13].

Given points $N_1, \ldots, N_q$ in E_2, some of which may not be distinct, and nonnegative weights $v_1, \ldots, v_q$, at least one of which is positive, it should be clear that the problem

$$\text{minimize } g(X)=\sum_{i=1}^{q} v_i|X-N_i|$$

can always be reduced to a General Fermat Problem by deleting any term for which v_i is zero, and by grouping together terms for which the points N_i are the same. The reduced form of the problem will be represented as follows: minimize $\bar{g}(X)$, where $\bar{g}$ is obtained from g in the manner indicated.

2. THE EXISTENCE OF A SOLUTION

While the fact that a point exists which minimizes f is perhaps intuitive, the methodology needed to prove that a solution exists seems of interest; also a condition for the problem to be poorly formulated is identified.

Some definitions are first in order; NFj and EFi will be said to *have an exchange* whenever $w_{ji} > 0$, and to have no exchange otherwise; likewise, NFj and NFk will be said to *have an exchange* whenever $d_{jk} > 0$, or $d_{kj} > 0$, and to have no exchange otherwise. It will be assumed that each NFj has an exchange with at least one other NF, as otherwise the optimum location of NFj can be found by solving the problem minimize $f_j(X_j)$, which reduces to a General Fermat Problem.

NFj is said to be *chained* if there exists a sequence of distinct NF's $j, j_1, \ldots, j_p$ such that NF's j and j_1 have an exchange, NF's j_t and j_{t+1} have an exchange for $t=1, \ldots, p-1$, and NFj_p and some EFi_j have an exchange. When $j=j_p$, that is, NFj and EFi_j have an exchange, then NF_j is said to be *trivially chained.* NF's which are not chained are said to be *unchained.* It should be clear that if there is at least one unchained NF then there are at least two unchained NF's, and that chained and unchained NF's have no exchanges.

As might be suspected, a problem which includes unchained NF's is poorly formulated. In the extreme case when all NF's are unchained, then all $w_{ji}=0$, so that $f(X)=f_0(X)$, which can be minimized trivially by taking all $X_j=P$, where P is *any* point in the plane. If there is at least one chained NF, then there will be at least two; number the chained NF's $1, 2, \ldots, q$ and number the unchained NF's $q+1, \ldots, n$. Since chained and unchained NF's have no exchanges,

$$d_{jk}=0 \qquad \text{for } j=1, \ldots, q \qquad \text{and } k=q+1, \ldots, n;$$

thus

$$f(X)=\sum_{1\leqq j<k\leqq q} d_{jk}|X_j-X_k|+\sum_{j=1}^{q} f_j(X_j)+\sum_{q+1\leqq j<k\leqq n} d_{jk}|X_j-X_k|. \tag{2}$$

Again by taking $X_j=P$ for $j=q+1, \ldots, n$, where P is any point in the plane, the last sum in (2) is zero, so that the choice of P has no effect upon the cost incurred due to the location of the chained NF's; this clearly indicates a poorly formulated problem. Subsequently, therefore, it will be assumed that all NF's are chained.

Let $f(X_1, \ldots, X_n)=r$ and define $d(j, k)$ to be d_{jk} for $j<k$, and to be d_{kj} for $j>k$. Let trivially chained NF's be numbered $1, \ldots, p$; it then follows that NFj and some EFi_j have an exchange, so that

$$|X_j-Q_{i_j}| \leqq r/w_{ji_j}, \qquad j=1, \ldots, p$$

Likewise, for $j=p+1, \ldots, n$ the chaining assumption implies there exist distinct facilities with numbers $j, j_1, \ldots, j_q, i_j$ such that

$$|X_j-X_{j_1}| \leqq r/d(j, j_1), \tag{3}$$

$$|X_{j_t}-X_{j_{t+1}}| \leqq r/d\,(j_t, j_{t+1}), \qquad t=1, \ldots, q-1, \text{ and} \tag{4}$$

$$|X_{j_q}-Q_{i_j}| \leqq r/w_{j_q i_j}. \tag{5}$$

If the term $b\,(r, j, i_j)$ is defined to be the sum of the numbers on the right side of the inequalities (3), (4), and (5), then the use of these inequalities together with the repeated use of the triangle inequality establishes that

$$|X_j - Q_{i_j}| \leqq b(r, j, i_j), \qquad j = p+1, \ldots, n.$$

A direct consequence of the upper bounds on $|X_j - Q_{i_j}|$ for $j=1, \ldots, n$ is the following remark.

REMARK 1: If the chaining assumption is made, and if r is any value of f, then the set $T(r) = \{X \epsilon E_{2n} : f(X) \leqq r\}$ is bounded.

The above remark is useful in establishing the following property.

PROPERTY 1: If S is any closed, nonempty subset of E_{2n}, then there exists at least one point X^* in S such that

$$f(X^*) = \min_{X \epsilon S} f(X).$$

PROOF: Let r be a value of f taken on for a point in S and define the set $S(r) = \{X \epsilon S : f(X) \leqq r\} = S \cap T(r)$. The continuity of f and the definition of $T(r)$ imply $T(r)$ is closed; since $S(r)$ is the intersection of two closed sets, $S(r)$ is also closed. Further, $S(r)$ is a subset of $T(r)$ and so is bounded by Remark 1. Thus $S(r)$ is a compact set, so the extreme value theorem implies there exists $X^* \epsilon\, S(r)$, such that

$$f(X^*) = \min_{X \epsilon S(r)} f(X).$$

The conclusion now follows from the definition of $S(r)$.

Note that when S is taken to be E_{2n}, Property 1 asserts that f has a minimum in E_{2n}. Property 1 may also be useful for studying the problem (1) when additional constraints involving X are attached to the problem.

3. PROPERTIES OF A SOLUTION

In this section it is shown that, when all NF's are chained, the optimum location of each NF is in the convex hull of the locations of the EF's. Also necessary conditions for a point X^* to minimize f are obtained.

REMARK 2: Let $X^* = (X_1^*, \ldots, X_n^*)$ minimize f, and, for $t = 1, \ldots, n$, define the function $g_t(X_t)$ on E_2 by

(6) $$g_t(X_t) = \sum_{1 \leqq j < t} d_{jt} |X_j^* - X_t| + \sum_{t < k \leqq n} d_{tk} |X_t - X_k^*| + f_t(X_t).$$

Then X_t^* minimizes g_t for $t = 1, \ldots, n$.

PROOF: For $t = 1, \ldots, n$, define

$$h_t(X_1^*, \ldots, X_{t-1}^*, X_{t+1}^*, \ldots, X_n^*) = f_0(X^*) - g_t(X_t^*) + f_t(X_t^*),$$

so that

(7) $$f(X^*) = g_t(X_t^*) + \left[h_t(X_1^*, \ldots, X_{t-1}^*, X_{t+1}^*, \ldots, X_n^*) + \sum_{j \neq t} f_j(X_j^*) \right].$$

Now if X_t^* does not minimize g_t then there exists $\hat{X}_t$ such that $g_t(\hat{X}_t) < g_t(X_t^*)$, but then the identity (7) clearly implies X^* does not minimize f, which gives a contradiction. Thus X_t^* minimizes g_t.

Subsequently, if S is any collection of a finite number of points in the plane, the convex hull of S will be denoted by $H(S)$. As a convenience, let $T = \{Q_1, \ldots, Q_m\}$, so that $H(T)$ denotes the convex hull of the locations of the EF's.

REMARK 3: Let $X^* = (X_1^*, \ldots, X_n^*)$ minimize f. Then, for $j=1, \ldots, n$,

$$X_j^* \in H(\{X_1^*, \ldots, X_{j-1}^*, X_{j+1}^*, \ldots, X_n^*\} \cup T).$$

PROOF: By Remark 2, X_j^* solves min $g_t(X_t)$, and so solves min $\bar{g}_t(X_t)$, the reduced General Fermat Problem; Kuhn's Result 1 thus implies X_j^* belongs to the convex hull of a subset of $\{X_1^*, \ldots, X_{j-1}^*, X_{j+1}^*, \ldots, X_n^*\} \cup T$, from which the conclusion follows.

REMARK 4: Let $X^* = (X_1^*, \ldots, X_n^*)$ minimize f. If the X_j^* are distinct, then, for $j=1, \ldots, n$,

$$X_j^* \in H(T).$$

PROOF: The proof is an immediate consequence of Remark 3, the hypothesis that the X_j^* are distinct, and the convex hull lemma of the appendix.

Notice that Remark 4 requires the restrictive assumption that the X_j^* are distinct; it will now be established that this assumption may be relaxed.

PROPERTY 2: Let $X^* = (X_1^*, \ldots, X_n^*)$ minimize f. Then, for $j=1, \ldots, n$,

$$X_j^* \in H(T).$$

PROOF: The proof will be by induction on n. Let (X_1^*, X_2^*) minimize

$$f(X_1, X_2) = d_{12}|X_1 - X_2| + f_1(X_1) + f_2(X_2).$$

If X_1^* and X_2^* are distinct, then the conclusion follows from Remark 4. If $X_1^* = X_2^*$, then clearly (X_1^*, X_2^*) solves the more constrained problem minimize $f(X_1, X_2)$ subject to $X_1 = X_2$, so that X_1^* solves the following problem which is equivalent to the one just stated:

$$\text{minimize } e(X_1) = \sum_{i=1}^{m} (w_{1i} + w_{2i})|X_1 - Q_i|.$$

By the chaining assumption, $w_{1i} + w_{2i} > 0$ for at least one i; hence X_1^* solves min $\bar{e}(X_1)$, the reduced General Fermat Problem, so Kuhn's Result 1 implies X_1^* is in the convex hull of some subset of T; thus $X_2^* = X_1^* \in H(T)$.

Now assume the property is true for the case $n = t$ and let $(X_1^*, \ldots, X_{t+1}^*)$ minimize $f(X_1, \ldots, X_{t+1})$. If the X_j^* are distinct, then the conclusion follows from Remark 4. If the X_j^* are not distinct, it may be assumed, without loss of generality, that $X_t^* = X_{t+1}^*$. Thus $(X_1^*, \ldots, X_{t+1}^*)$ solves the problem minimize $f(X_1, \ldots, X_{t+1})$ subject to $X_t = X_{t+1}$, so that $(X_1^*, \ldots, X_t^*)$ solves the following problem which is equivalent to the one just stated:

$$\text{minimize } e\ (X_1, \ldots, X_t) \equiv \sum_{1 \leq j < k \leq t-1} d_{jk}|X_j - X_k| + \sum_{j=1}^{t-1} (d_{jt} + d_{jt+1})|X_j - X_t| + \sum_{j=1}^{t-1} f_j\ (X_j) + \sum_{i=1}^{m} (w_{ti} + w_{t+1i})|X_t - Q_i|.$$

But the latter problem has the same form as f for $n=t$, and so the inductive assumption implies $X_j^* \epsilon H(T)$ for $j=1, \ldots, t$. Since $X_{t+1}^* = X_t^*$, the conclusion now follows.

Next it is shown that Kuhn's necessary conditions for a solution to the General Fermat Problem can be extended to provide necessary conditions for a solution to the problem of interest. The question of whether the conditions are also sufficient remains an open one.

PROPERTY 3: Let $X^* = (X_1^*, \ldots, X_n^*)$ minimize f. Then

$$RK\ (X^*) \equiv (R_1\ (X_1^*), \ldots, R_n\ (X_n^*)) = 0,$$

where $R_j\ (X_j^*)$ is Kuhn's modified negative gradient vector $R\ (X^*)$ as it applies to the reduced General Fermat Problem minimize $\bar{g}_t\ (X_t)$, where g_t is defined by (6).

PROOF: From Remark 2, for $t=1, \ldots, n$ X_t^* minimizes $g_t\ (X_t)$, and so minimizes $\bar{g}_t\ (X_t)$, the reduced General Fermat form; Kuhn's Result 2 then implies $R_t\ (X_t^*) = 0$, so that $RK\ (X^*) = 0$.

4. CONDITIONS FOR THE FUNCTION f TO BE STRICTLY CONVEX

As pointed out by Love [13], f is convex on E_{2n}; the value of knowing that f is strictly convex, of course, lies in the fact that it has a unique minimum. It seems an interesting fact that the conditions for strict convexity do not depend in any manner upon f_0.

REMARK 5: For $j=1, \ldots, n$ let the set $S_j = \{Q_i : w_{ji} > 0\}$ be nonempty and suppose the points in S_j are not collinear; that is, let each NFj have exchanges with three or more EF's which are not collinear. Then f is strictly convex on E_{2n}.

PROOF: Kuhn's Result 3 and the hypotheses imply the reduced General Fermat form of $f_j(X_j)$, $\bar{f}_j(X_j)$, is strictly convex on E_2, and so

$$\sum_{j=1}^{n} f_j(X_j) = \sum_{j=1}^{n} \bar{f}_j(X_j)$$

is strictly convex on E_{2n}. The use of the triangle inequality establishes that f_0 is convex on E_{2n}. Thus f is the sum of a convex and a strictly convex function, both having domain E_{2n}, and is thus strictly convex on E_{2n}.

REMARK 6: Let f be strictly convex on E_{2n}. Then, for $j=1, \ldots, n$, the set $S_j = \{Q_i : w_{ij} > 0\}$ is nonempty, and the points in the set are not collinear.

PROOF: Suppose first that at least one set S_j is empty; without loss of generality, suppose S_n is empty; let $d = d_{1n} + \ldots + d_{n-1n}$. Then

(8) $$f(0, \ldots, 0, X_n) = d|X_n| + \sum_{j=1}^{n-1} f_j(0).$$

As noted by Kuhn, the function $d|X_n|$ will be linear on any half-line H in E_2 ending at the origin. If one chooses two distinct points say X_n^1, X_n^2 in H, the expression (8) then implies f will be linear on the line segment joining $(0, \ldots, 0, X_n^1)$ and $(0, \ldots, 0, X_n^2)$ in E_{2n}, which contradicts the fact that f is strictly convex. Thus each set S_j is nonempty.

Next suppose that the points in at least one set S_j are collinear; without loss of generality, suppose the points in S_n are collinear, and that $w_{n1} > 0$. Define

$$\hat{w}_{n1} = \sum_{j=1}^{n-1} d_{jn} + w_{n1},$$

$$\hat{w}_{ni} = w_{ni}, \, i = 2, \ldots, m,$$

and

$$\hat{f}(X_n) = \sum_{i=1}^{m} \hat{w}_{ni} |X_n - Q_i|.$$

Then

$$f(Q_1, \ldots, Q_1, X_n) = \sum_{j=1}^{n-1} f_j(Q_1) + \hat{f}(X_n). \tag{9}$$

By Kuhn's Result 3, $\hat{f}$ is piecewise linear on the line L in E_2 containing S_n, so distinct points X_n^1, X_n^2 in L may be chosen such that $\hat{f}$ is linear on the line segment in E_2 joining the points X_n^1 and X_n^2. The expression (9) then implies f will be linear on the line segment joining $(Q_1, \ldots, Q_1, X_n^1)$ and $(Q_1, \ldots, Q_1, X_n^2)$ in E_{2n}, which again contradicts the fact that f is strictly convex. Thus the points in each set S_j are not collinear.

Remarks 5 and 6 may be summarized to give the following:

PROPERTY 4: The function f is strictly convex if, and only if, for $j = 1, \ldots, n$, the set $S_j = \{Q_i : w_{ji} > 0\}$ is nonempty, and the points in each set S_j are not collinear.

5. A DUAL TO THE PROBLEM

A dual to the problem will now be obtained. Complementary slackness conditions similar to those of linear programming will be developed; obvious changes in the wording of the complementary slackness conditions give necessary and sufficient conditions for a point in E_{2n} to minimize f, and for a feasible solution to the dual problem to be a maximum feasible solution. Also the vectors of the dual problem have interesting interpretations in terms of the location problem. The duality relationships make the dual problem of substantial theoretical interest; whether or not the dual problem can be directly useful computationally in solving the location problem is still an open question.

In order to obtain a dual to the problem, use is made of the duality results of Sinha [19] and [20], as extended by Bhatia and Kaul [2], Bhatia [3], and Mond [15]. The versions of Sinha's primal and dual problems needed are stated as follows:

PRIMAL:

$$\text{minimize } F(S) \equiv c^tS + \sum_{i=1}^{m} (S^tD_iS)^{1/2}$$

$$\text{subject to } AS = b;$$

DUAL:

$$\text{maximize } G(Y) \equiv b^tY$$

$$\text{subject to } A^tY - \sum_{i=1}^{m} D_iZ_i = c$$

$$Z_i^tD_iZ_i \leqq 1, \qquad i = 1, \ldots, m.$$

The above forms of the primal and dual problems are equivalent to the forms found in [20] and [3], and are essentially special cases of the forms found in [15]. The equivalence is not entirely obvious, however, due to the inclusion of the quadratic forms; an algebraic proof of the equivalence, as opposed to the geometric approach in [15], is included in the appendix.

In the above problems, A may be assumed to be a given p by q matrix of real numbers, b and c are given p by 1 and q by 1 vectors, S is a q by 1 vector of variables, $Z_1, \ldots, Z_m$ are q by 1 vectors of variables, Y is a p by 1 vector of variables, and the q by q matrices $D_1, \ldots, D_m$ are given and are positive semidefinite.

Three theorems relating the primal and dual problems are of interest:

THEOREM 1: For any feasible solution S to the primal problem and any feasible solution $(Y, Z_1, \ldots, Z_m)$ to the dual problem, $F(S) \geqq G(Y)$.

THEOREM 2: If S^* is a minimum feasible solution to the primal problem, then there exists a maximum feasible solution $(Y^*, Z_1^*, \ldots, Z_m^*)$ to the dual problem, and $F(S^*) = G(Y^*)$.

THEOREM 3: Suppose the dual problem satisfies the Kuhn-Tucker constraint qualification [9], [14]. If $(Y^*, Z_1^*, \ldots, Z_m^*)$ is a maximum feasible solution to the dual problem, then there exists a minimum feasible solution S^* to the primal problem, and $F(S^*) = G(Y^*)$.

The first theorem is due to Sinha [20], while the second and third theorems are due to Bhatia [3] and Mond [15], respectively. Sinha originally proved Theorems 2 and 3 under the assumption that the set of all feasible solutions to the primal problem is bounded. The basic dual inequalities in both Kuhn's and Sinha's work are versions of the Cauchy-Schwartz inequality; Kuhn's duality results appear to be a previously unrecognized predecessor of Sinha's.

The approach used to obtain the dual to the location problem (1) will be to transform it into a special case of Sinha's primal problem, and then construct and simplify the resultant dual problem.

Defining the vectors $U_{jk} = X_j - X_k$ for $1 \leqq j < k \leqq n$ and $V_{ji} = X_j - Q_i$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, gives the following problem, which is clearly equivalent to the location problem and is a special case of Sinha's primal problem:

$$\text{minimize} \sum_{1 \leqq j < k \leqq n} d_{jk}(U_{jk}^tU_{jk})^{1/2} + \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji}(V_{ji}^tV_{ji})^{1/2}$$

$$\text{subject to } X_j - X_k - U_{jk} = 0, \qquad 1 \leqq j < k \leqq n$$

$$X_j - V_{ji} = Q_i, \qquad i = 1, \ldots, m \text{ and } j = 1, \ldots, n.$$

The derivation of the dual of the above problem is given in the appendix, and constitutes the proof of Property 5.

In the dual to the location problem stated in Property 5 below, the variables are the entries in the 2 by 1 vectors T_{jk} and Y_{ji}, and it should be understood that sums extending from 1 to 0 or from $n+1$ to n are defined to be the zero vector.

PROPERTY 5: Call the location problem (1) the primal problem and the following problem the dual of the location problem:

$$\text{maximize } g(Y) \equiv \sum_{j=1}^{n} \sum_{i=1}^{m} Q_i^t Y_{ji}$$

$$\text{subject to} - \sum_{j=1}^{s-1} T_{js} + \sum_{k=s+1}^{n} T_{sk} + \sum_{i=1}^{m} Y_{si} = 0, \qquad s = 1, \ldots, n$$

$$|T_{jk}| \leqq d_{jk}, \qquad 1 \leqq j < k \leqq n$$

$$|Y_{ji}| \leqq w_{ji}. \qquad i = 1, \ldots, m \text{ and}$$

$$j = 1, \ldots, n.$$

Then Theorems 1 through 3 apply to the primal location problem and its dual on replacing $F(S)$ by $f(X)$, $G(Y)$ by $g(Y)$, and considering X and $(T, Y_1, \ldots, Y_n) \equiv (T_{12}^t \ldots T_{n-1n}^t Y_{11}^t \ldots Y_{1m}^t \ldots Y_{n1}^t \ldots Y_{nm}^t)^t$ to be feasible solutions to the primal and dual problems, respectively.

Kuhn's dual is obtained by taking $n=1$, so that the T vectors no longer appear. The above dual also extends the linear programming dual given in [4] for the case where (1) is defined using rectilinear distances, and, in fact, includes that dual as a special case when each 2 by 1 vector is interpreted to be a number and each Euclidian norm is interpreted to be an absolute value.

It may be of interest to note when additional linear constraints in X are attached to the location problem (1) that a dual problem can still be obtained; the additional linear constraints are simply attached to the problem equivalent to (1) stated prior to Property 5, and then the appropriate form of Sinha's primal and dual problems are again used. Also, Sinha discusses briefly a method of solving his primal problem, given a maximum feasible solution to his dual problem. An alternative method which takes advantage of the special structure of the dual of the location problem will be examined subsequently.

In a manner similar to that of References [12] and [26], insight into the location problem (1) may be obtained by studying its dual and vice versa. Equations (10) through (13) below will be called the *complementary slackness conditions*.

PROPERTY 6: Let $X = (X_1, \ldots, X_n) \epsilon E_{2n}$ and let $(T, Y_1, \ldots, Y_n)$ by any feasible solution to the dual problem. Then X minimizes f and $(T, Y_1, \ldots, Y_n)$ is a maximum feasible solution to the dual problem if and only if

(10) $$(Q_i - X_j)^t Y_{ji} = |Y_{ji}| \;\; |X_j - Q_i|, \qquad i=1, \ldots, m \text{ and } j=1, \ldots, n$$

(11) $$(X_k - X_j)^t T_{jk} = |T_{jk}| \;\; |X_j - X_k|, \qquad 1 \leqq j < k \leqq n$$

(12) $$(|Y_{ji}| - w_{ji})|X_j - Q_i| = 0, \qquad i=1, \ldots, m \text{ and } j=1, \ldots, n$$

(13) $$(|T_{jk}| - d_{jk})|X_j - X_k| = 0, \qquad 1 \leqq j < k \leqq n.$$

PROOF:

(a) Let $X \epsilon E_{2n}$ and let $(T, Y_1, \ldots, Y_n)$ be any feasible solution to the dual problem; it will first be shown that $g(Y) \leqq f(X)$:

$$g(Y) = \sum_{i=1}^{m} \sum_{s=1}^{n} Q_i^t Y_{si} - \sum_{s=1}^{n} X_s^t \left(- \sum_{j=1}^{s-1} T_{js} + \sum_{k=s+1}^{n} T_{sk} + \sum_{i=1}^{m} Y_{si} \right)$$
$$= \sum_{s=1}^{n} \sum_{i=1}^{m} (Q_i - X_s)^t Y_{si} + \sum_{1 \leqq j < k \leqq n} (X_k - X_j)^t T_{jk}.$$

By the Schwartz inequality, for $s=1, \ldots, n$ and $i=1, \ldots, m$,

(14) $$(Q_i - X_s)^t Y_{si} \leqq |Q_i - X_s| \;\; |Y_{si}| = |Y_{si}| \;\; |X_s - Q_i|;$$

also, for $1 \leqq j < k \leqq n$,

(15) $$(X_k - X_j)^t T_{jk} \leqq |X_k - X_j| \;\; |T_{jk}| = |T_{jk}| \;\; |X_j - X_k|.$$

Further, for $i=1, \ldots, m$ and $s=1, \ldots, n$,

(16) $$|Y_{si}| \;\; |X_s - Q_i| \leqq w_{si} |X_s - Q_i|$$

while, for $1 \leqq j < k \leqq n$,

(17) $$|T_{jk}| \;\; |X_j - X_k| \leqq d_{jk} |X_j - X_k|.$$

The inequalities (14) through (17) now imply that $g(Y) \leqq f(X)$. (Note that this part of the proof is Sinha's Theorem 1 as it applies to this problem; the proof also follows Kuhn's proof of the basic dual inequality for the General Fermat Problem.)

(b) Clearly X minimizes f and $(T, Y_1, \ldots, Y_n)$ is a maximum feasible solution to the dual if $g(Y) = f(X)$; when the complementary slackness conditions hold, the inequalities (14) through (17) hold as equalities, so that indeed $g(Y) = f(X)$ and so the complementary slackness conditions imply the optimality of X and $(T, Y_1, \ldots, Y_n)$.

(c) Let X minimize f and $(T, Y_1, \ldots, Y_n)$ be a maximum feasible solution to the dual; then, by Property 5, $g(Y) = f(X)$. In part (a) it was established that (10) through (13) still hold when each equality sign is replaced by an equal to or less than sign; if any of the inequalities held strictly, the

same approach as in part (a) would imply that $g(Y) < f(X)$, which would contradict Property 5. Thus the complementary slackness conditions hold.

Useful information can easily be obtained from the complementary slackness conditions. Let $(X_1, \ldots, X_n)$ minimize f and let $(T, Y_1, \ldots, Y_n)$ be a maximum feasible solution to the dual problem. From (12), $|Y_{ji}| < w_{ji}$ implies $X_j = Q_i$, while $X_j \neq Q_i$ implies $|Y_{ji}| = w_{ji}$; since the points $Q_1, \ldots, Q_m$ are distinct, it follows that at most one of the inequalities $|Y_{j1}| \leqq w_{j1}, |Y_{j2}| \leqq w_{j2}, \ldots, |Y_{jm}| \leqq w_{jm}$ can hold strictly, for $j = 1, \ldots, n$. Likewise, from (13), $|T_{jk}| < d_{jk}$ implies $X_j = X_k$, while $X_j \neq X_k$ implies $|T_{jk}| = d_{jk}$. Further, it is an interesting fact, when $X_j \neq Q_i$, that (10) is equivalent to

$$Y_{ji} = w_{ji}(Q_i - X_j)/|X_j - Q_i|; \tag{18}$$

the equivalence is due to the readily established fact that (18) gives the unique solution to the system of equations $|Y_{ji}| = w_{ji}$, and (10) with $|Y_{ji}|$ replaced by w_{ji}. Likewise, when $X_j \neq X_k$, then (11) is equivalent to

$$T_{jk} = d_{jk}(X_k - X_j)/|X_j - X_k|. \tag{19}$$

The expressions (18) and (19) are of particular interest. Assume again that $X = (X_1, \ldots, X_n)$ minimizes f and $(T, Y_1, \ldots, Y_n)$ is a maximum feasible solution to the dual problem. Assume further that the locations of all facilities are distinct, so that (18) and (19) are certain to hold. Equation (18) implies geometrically that when $Y_{ji} \neq 0$ the line through Q_i parallel to Y_{ji} passes through X_j, while Equation (19) implies geometrically that when $T_{jk} \neq 0$ the line through X_k parallel to T_{jk} passes through X_j. Thus, as observed by Kuhn, and by Witzgall and Rockafellar, given vectors Y_{js} and Y_{jt}, neither of which is a scalar multiple of the other, the lines through Q_s and Q_t parallel to Y_{js} and Y_{jt}, respectively, intersect at X_j; this fact is useful in constructing a solution to (1) given a maximum feasible solution to the dual problem. In a similar manner, once some of the points X_j have been determined using (18), then Equation (19) may be used to find others. Thus a maximum feasible solution to the dual is quite useful in finding a point $X = (X_1, \ldots, X_n)$ which minimizes f; the complementary slackness conditions provide a convenient computational check, once the point is found, for it to minimize f.

Further information may be obtained from Equations (18) and (19) under the assumptions of the previous paragraph. A direct computation shows that Y_{ji} as defined by (18) is just the vector of partial derivatives of f_j with respect to the entries of Q_i, so that $\sum_{j=1}^{n} Y_{ji}$ is the vector of partial derivatives of f with respect to the entries of Q_i. The vector $-\sum_{j=1}^{n} Y_{ji}$ thus gives the direction of steepest descent in which to move Q_i in order to reduce the minimum value of f. Likewise, a direct computation establishes that T_{jk} as defined by (19) is the vector of partial derivatives of the term $d_{jk}|X_j - X_k|$ with respect to the entries of X_k. Thus the vectors T_{jk} also have an interpretation in terms of the primal problem, although perhaps not as interesting a one as the interpretation of the vectors Y_{ji}. These interpretations of the dual vectors correspond closely to those of the dual variables in linear programming.

Finally, as noted by Sinha, his dual problem is a convex programming problem which satisfies all the customary differentiability assumptions. Thus, in principle, a number of convex programming algorithms are available for solving the dual of the location problem; an algorithm which takes advantage

of the special structure of the dual problem still remains to be developed. That the development of such an algorithm is possible, however, is strongly suggested by the experience reported in [4].

Appendix

6. CONVEX HULL LEMMA

If S is any collection of a finite number of points in E_r, the convex hull of S will be denoted by $H(S)$. Use will often be made of the fact that $S' \subset H(S)$ implies $H(S' \cup S) = H(S)$.

LEMMA: Let S be any collection of a finite number of points $\{P_i\}$ in E_r. If $Z_1, \ldots, Z_n$ are any n distinct points in E_r and if

(1) $$Z_j \epsilon H(\{Z_1, \ldots, Z_n\} - \{Z_j\}) \cup S) \qquad \text{for } j = 1, \ldots, n$$

then

(2) $$Z_j \epsilon H(S) \qquad \text{for } j = 1, \ldots, n.$$

PROOF: The proof will be by induction on n. Let $n = 2$; using (1) to express Z_1 and Z_2 as convex combinations gives

(3) $$Z_1 = a_1 Z_2 + \sum_i c_{i1} P_i$$

(4) $$Z_2 = a_2 Z_1 + \sum_i c_{i2} P_i.$$

If $a_2 = 0$ then $Z_2 \epsilon H(S)$ so $Z_1 \epsilon H(\{Z_2\} \cup S)$ implies $Z_1 \epsilon H(S)$. If $a_2 > 0$, then solving (4) for Z_1 and substituting into (3) gives, after rearranging terms,

$$(1 - a_1 a_2) Z_2 = \sum_i (a_2 c_{i1} + c_{i2}) P_i.$$

Since $Z_1 \neq Z_2$, $a_1 < 1$ and $a_2 < 1$, so that $1 - a_1 a_2 > 0$. Thus $Z_2 \epsilon H(S)$ if

$$\sum_i (a_2 c_{i1} + c_{i2})/(1 - a_1 a_2) = 1,$$

i.e.,

$$1 - a_1 a_2 = \sum_i a_2 c_{i1} + \sum_i c_{i2} = a_2(1 - a_1) + (1 - a_2),$$

which is true. Further, $Z_1 \epsilon H(\{Z_2\} \cup S) = H(S)$, which completes the proof for $n = 2$.

Now make the inductive assumption that the claim holds for $n = p - 1$. For any $t \epsilon \{1, 2, \ldots, p\}$, let $S_t = \{Z_t\} \cup S$. By hypothesis,

$$Z_j \epsilon H((\{Z_1, \ldots, Z_{t-1}, Z_{t+1}, \ldots, Z_p\} - \{Z_j\}) \cup S_t), \qquad j = 1, \ldots, p$$

so the inductive assumption implies

$$Z_j \epsilon H(S_t) = H(\{Z_t\} \cup S), \qquad j = 1, \ldots, p.$$

In particular, with $t = r$ and $j = q$, $r \neq q$,

$$Z_q \epsilon H(\{Z_r\} \cup S),$$

while, with $t = q$ and $j = r$, $r \neq q$

$$Z_r \epsilon H(\{Z_q\} \cup S).$$

Thus, the case $n = 2$ implies $Z_r, Z_q \epsilon H(S)$ for any r, q such that $1 \leqq r \leqq p$, $1 \leqq q \leqq p$. Q.E.D.

7. EQUIVALENT VERSIONS OF SINHA'S PROBLEMS

One version of Sinha's problems which is obviously equivalent to the version found in [3] or [20] is as follows:

PRIMAL:

$$\text{minimize } F(S)$$
$$\text{subject to } AS \geqq b,\ S \geqq 0;$$

DUAL:

$$\begin{aligned} &\text{maximize } G(Y) \\ &\text{subject to } A^t Y - \sum_{i=1}^{m} D_i Z_i \leqq c \\ &\qquad\qquad Z_i^t D_i Z_i \leqq 1, \qquad i = 1, \ldots, m \\ &\qquad\qquad Y \geqq 0. \end{aligned}$$

Call the above primal and dual problems $P1$ and $D1$, respectively, and the primal and dual problems of section 5 $P0$ and $D0$, respectively. The key to showing the equivalence of $P0$ and $D0$ to $P1$ and $D1$ is the following identity: for any quadratic form $X^t DX$, where D is n by n,

$$(X_1^t - X_2^t)D(X_1 - X_2) = (X_1^t X_2^t) \begin{bmatrix} D & -D \\ -D & D \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{11}$$

for all $X_1, X_2 \epsilon E_n$.

On making the change of variables $S = S_1 - S_2$ in $P0$, with $S_1 \geqq 0, S_2 \geqq 0$, defining

$$\bar{c}^t = (c^t, -c^t),\ \bar{b}^t = (b^t, -b^t),\ \bar{S}^t = (S_1^t S_2^t)$$

$$\bar{A} = \begin{bmatrix} A & -A \\ -A & A \end{bmatrix}, \qquad \bar{D}_i = \begin{bmatrix} D_i & -D_i \\ -D_i & D_i \end{bmatrix}$$

and using the identity (11), $P0$ may be written as

$$\text{minimize } \bar{c}^t\bar{S} + \sum_{i=1}^{m} (\bar{S}^t\bar{D}_i\bar{S})^{1/2}$$

$$\text{subject to } \bar{A}\bar{S} \geqq \bar{b}, \bar{S} \geqq 0.$$

The latter problem is in the form of $P1$, so that its dual is given by

$$\text{maximize } \bar{b}^t\bar{Y}$$

$$\text{subject to } \bar{A}^t\bar{Y} - \sum_{i=1}^{m} \bar{D}_i\bar{Z}_i \leqq \bar{c}$$

$$\bar{Z}_i^t\bar{D}_i\bar{Z}_i \leqq 1, \qquad i = 1, \ldots, m$$

$$\bar{Y} \geqq 0.$$

On partitioning $\bar{Y}$ and $\bar{Z}_i$ as follows:

$$\bar{Y}^t = (Y_1^tY_2^t), \qquad \bar{Z}_i^t = ((Z_1^i)^t(Z_2^i)^t),$$

again using the identity (11), and letting $Y = Y_1 - Y_2$, $Z_i = Z_1^i - Z_2^i$, the latter problem readily reduces to $D0$.

8. PROOF OF PROPERTY 5

In order to put the constrained optimization problem equivalent to the location problem (1) into matrix form, define the vectors U, V_j, X, and Q by

$$U^t = (U_{12}^t \ldots U_{1n}^tU_{23}^t \ldots U_{2n}^t \ldots U_{n-1n}^t),$$

$$V_j^t = (V_{j1}^t \ldots V_{jm}^t), \qquad j = 1, \ldots, n$$

$$X^t = (X_1^t \ldots X_n^t)$$

$$Q^t = (Q_1^t \ldots Q_m^t).$$

Let $p = n(n-1)/2$ and let B be the $2p$ by $2n$ matrix, such that

$$(BX)^t = (X_1^t - X_2^t \ldots X_1^t - X_n^tX_2^t - X_3^t \ldots X_2^t - X_n^t \ldots X_{n-1}^t - X_n^t).$$

Let H_t be the $2m$ by $2n$ matrix whose entries are all zero except for columns $2t$ and $2t-1$, which con-

sist of 2 by 2 identity matrices, $t=1, \ldots, n$. Then the constraints of the above problem may be written as

$$\begin{bmatrix} B-I_{2p} & & & & \\ H_1 & -I_{2m} & & & \\ H_2 & & -I_{2m} & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ H_n & & & & -I_{2m} \end{bmatrix} \begin{bmatrix} X \\ U \\ V_1 \\ \cdot \\ \cdot \\ \cdot \\ V_n \end{bmatrix} = \begin{bmatrix} 0 \\ Q \\ Q \\ \cdot \\ \cdot \\ \cdot \\ Q \end{bmatrix}$$

which defines A, S, and b of Sinha's primal problem; note that A is a $2(p+mn)$ by $2(n+p+mn)$ matrix. Clearly the c vector of Sinha's problem is the zero vector, and matrices, say D^0_{jk} and D_{ji} may be defined so that

$$S^t D^0_{jk} S = U^t_{jk} d^2_{jk} U_{jk}, \qquad 1 \leqq j < k \leqq n \tag{12}$$

$$S^t D_{ji} S = V^t_{ji} w^2_{ji} V_{ji}, \qquad i=1, \ldots, m \text{ and } j=1, \ldots, n. \tag{13}$$

Each D matrix has $2(n+p+mn)$ rows and columns, and is a zero matrix with the exception of two adjacent elements of d^2_{jk} or w^2_{ji} placed along the diagonal so as to satisfy Equations (12) or (13), respectively. The objective function of Sinha's problem is now the sum of the square roots of the quadratic forms appearing on the left side of Equations (12) and (13).

In order to construct the dual problem, first define the vectors T, $Y_1, \ldots, Y_n$, and Y by

$$T^t = (T^t_{12} \ \ldots \ T^t_{1n} T^t_{23} \ \ldots \ T^t_{2n} \ \ldots \ T^t_{n-1n})$$

$$Y^t_j = (Y^t_{j1} \ \ldots \ Y^t_{jm}), \qquad j=1, \ldots, n$$

$$Y^t = (T^t Y^t_1 \ \ldots \ Y^t_n).$$

The objective function of the dual problem is then

$$b^t Y = 0^t T + \sum_{j=1}^{n} Q^t Y_j = \sum_{j=1}^{n} \sum_{i=1}^{m} Q^t_i Y_{ji},$$

which gives the desired dual objective function. Represent the Z vectors of Sinha's dual problem by

$\hat{Z}^0_{jk}$, $1 \leqq j < k \leqq n$, and by $\hat{Z}_{ji}$ for $i=1, \ldots, m$ and $j=1, \ldots, n$. Then it can be shown that

$$\left(\sum_{1\leqq j<k\leqq n} D^0_{jk}\hat{Z}^0_{jk}\right)^t = (0^t d^2_{12} Z^{0t}_{12} \ \ldots \ d^2_{1n} Z^{0t}_{1n} d^2_{23} Z^{0t}_{23} \ \ldots \ d^2_{2n} Z^{0t}_{2n} \ \ldots \ d^2_{n-1n} Z^{0t}_{n=1n} 0^t),$$

where, in the expression on the right hand side the first zero matrix is $2n$ by 1 and the second zero matrix is $2(mn)$ by 1. Further,

$$\left(\sum_{j=1}^{n}\sum_{i=1}^{m} D_{ji}\hat{Z}_{ji}\right)^t = (0^t w^2_{11} Z^t_{11} \ \ldots \ w^2_{1m} Z^t_{1m} \ \ldots \ w^2_{n1} Z^t_{n1} \ \ldots \ w^2_{nm} Z^t_{nm}),$$

where the zero matrix in the expression on the right hand side is $2(n+p)$ by 1.

By constructing the dual according to Sinha and simplifying we get the constraints

(14) $$-\sum_{j=1}^{s-1} T_{js} + \sum_{k=s+1}^{n} T_{sk} + \sum_{i=1}^{m} Y_{si} = 0, \qquad s=1, \ldots, n$$

(15) $$-T_{jk} - d^2_{jk} Z^0_{jk} = 0, \qquad 1 \leqq j < k \leqq n$$

(16) $$-Y_{ji} - w^2_{ji} Z_{ji} = 0, \qquad i=1, \ldots, m \text{ and } j=1, \ldots, n$$

(17) $$Z^{0t}_{jk} d^2_{jk} Z^0_{jk} \leqq 1, \qquad 1 \leqq j < k \leqq n$$

(18) $$Z^t_{ji} w^2_{ji} Z_{ji} \leqq 1, \qquad i=1, \ldots, m \text{ and } j=1, \ldots, n.$$

Further simplifications are possible; when $d_{jk} > 0$, then solving (15) for Z^0_{jk} and substituting into (17) gives $T^t_{jk} T_{jk} \leqq d^2_{jk}$, or, equivalently,

(19) $$|T_{jk}| \leqq d_{jk}, \qquad 1 \leqq j < k \leqq n.$$

When $d_{jk} = 0$ then (17) is redundant and $T_{jk} = 0$, so (15) and (17) can again be replaced by (19). Likewise, (16) and (18) may be replaced by

(20) $$|Y_{ji}| \leqq w_{ji}, \qquad i=1, \ldots, m \text{ and } j=1, \ldots, n.$$

Thus the constraints of the dual of the location problem are given by (14), (19), and (20).

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A DUEL COMPLICATED BY FALSE TARGETS AND UNCERTAINTY AS TO OPPONENT TYPE

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ABSTRACT

An attacker, being one of two types, initiates an attack at some time in the interval $[-T, 0]$. The a priori probabilities of each type are known. As time elapses the defender encounters false targets which occur according to a known Poisson process and which can be properly classified with known probability. The detection and classification probabilities for each type attacker are given. If the defender responds with a weapon at the time of attack, he survives with a probability which depends on the number of weapons in his possession and on attacker type. If he does not respond, his survival probability is smaller. These probabilities are known, as well as the current number of weapons in the defender's possession. They decrease as the number of weapons decreases. The payoff is the defender's survival probability.

An iterative system of first-order differential equations is derived whose unique solution $V_1(t), V_2(t), \ldots, V_k(t), \ldots$ is shown to be the value of the game at time t, when the defender has $1, 2, \ldots, k, \ldots$ weapons, respectively. The optimal strategies are determined. Limiting results are obtained as $t \to -\infty$, while the ratio of the number of weapons to the expected number of false targets remaining is held constant.

INTRODUCTION

A duel between a *defender* and an *attacker* is to take place at some time t in the time interval $[-T, 0]$. The duel is initiated by the attacker who chooses the time of attack and knows the current weapon strength of the defender. The defender, having detected and classified the attacker, decides whether or not to respond with a weapon. The duel is complicated by the imperfect ability of the defender to detect and identify the type of attacker or to distinguish between the attacker and "false targets."

As a generalization of [3] we suppose the attacker is one of two types, type n or type c, and that the defender is able to estimate the a priori probabilities of each type. The type is selected according to this a priori distribution and remains fixed. The defender encounters false targets which occur according to the Poisson process with known density $\lambda(t)$ at time t. An obvious example is that of a transiting ship and an attacking submarine which is either nuclear or conventional.

The detection and classification capabilities are known. The ability of the defender to survive an attack depends upon the attacker type, the number of weapons in the defender's possession, and the time at which he reacts to an attack. If he has k weapons and reacts early (i.e., fires first) against a type n attacker, he survives with probability $p_k^{(n)}$. If he delays response (i.e., the type n attacker fires first), he survives with strictly smaller probability $q_k^{(n)}$. Against a type c attacker the corresponding probabilities are $p_k^{(c)}$ and $q_k^{(c)}$. There is an advantage to the defender to fire first; however, being too anxious to fire first may result in weapons being wasted against false targets, leading to a weakened condition since we assume these probabilities are nondecreasing functions of the number of weapons.

The attacker must select an attack time, allowing the time to depend upon attacker type and on the number of weapons currently in the possession of the defender. In the submarine example, this is a realistic assumption if the submarine is "shadowing" the transiting ship outside detection range and can detect the firing of a weapon. An optimal strategy for the attacker, whatever his type, takes maximum advantage of the defender's confusion as to attacker type and his tendency to fire at false targets.

The payoff is the defender's survival probability, including the possibility of attack at $t=0$.

We shall derive an iterative system of first-order differential equations and prove that its unique solution $V_1(t), V_2(t), \ldots, V_k(t), \ldots$ is the value of the game at time t when the defender has 1, 2, . . ., k, . . . weapons, respectively. The optimal strategies are determined. The limit of the value $V_k(t)$ is obtained as the number of weapons k tends to infinity, while the ratio of k to the expected number of false targets remaining is held constant. Strategies for the defender and the attacker are found which are "almost" optimal for this limiting process. This is of practical significance since, for cases in which k is large and/or the transit time is long, it is not necessary to solve the system of differential equations. The limiting value can be an important consideration in the decision of how many weapons to carry, or in measuring the net effect of improved detection capabilities at the expense of increased false-target rate, or in assessing the gain resulting from improved classification capabilities, etc.

THE MODEL

The following parameters are known by both players:

k = the current weapon strength for the defender.

$p_k^{(n)}$ = the probability of survival for the defender, given that he possesses k weapons and responds at the time of attack by a type n attacker, $k=1,2,\ldots$.

$q_k^{(n)}$ = the probability of survival for the defender, given that he possesses k weapons and does not respond until after an attack by a type n attacker, $k=1,2,\ldots$.

$p_k^{(c)}$ = the probability of survival for the defender, given that he possesses k weapons and responds responds at the time of attack by a type c attacker, $k=1,2,\ldots$.

$q_k^{(c)}$ = the probability of survival for the defender, given that he possesses k weapons and does not respond until after an attack by a type c attacker, $k=1,2,\ldots$.

D_n = the probability of detection, given attack by a type n attacker.

D_c = the probability of detection, given attack by a type c attacker.

$$\begin{pmatrix} c_{nn} & c_{nc} & c_{nf} \\ c_{cn} & c_{cc} & c_{cf} \\ c_{fn} & c_{fc} & c_{ff} \end{pmatrix} = \text{classification matrix, where}$$

c_{ij} = the probability of classifying the attacker as type j, given that the attacker is type i, $i=n$ or c; $j=n$, c or f, and c_{fj} = the probability of classifying a false target as type j.

$P(i)$ = the prior probability that the attacker is type i, $i=n$ or c. The attacker also knows his type.

$\lambda(t)$ = the density of false-target detection at time t.

It is given that $p_k^{(n)} > q_k^{(n)}$, $p_k^{(c)} > q_k^{(c)}$ and each of these parameters is a nondecreasing function of k. The dependence on k reflects the usual circumstance that the defender's survival probability is a func-

tion of the number of weapons in his possession. The probability of two or more false-target detections in any interval of length Δt is $o(\Delta t)$, and the probability of a false-target detection in the interval $(t, t+\Delta t)$ is $\lambda(t)\Delta t+o(\Delta t)$.

A (behavior) strategy for the defender is a collection of triples $(x_n^{(k)}(t), x_c^{(k)}(t), x_f^{(k)}(t))$, $k=1,2,\ldots$, $t\epsilon[-T, 0]$, where, given k weapons and detection and classification at time t, these are the probabilities of firing when the classification is "type n," "type c," or "false target," respectively.

A strategy for the attacker is a collection of pairs $(a_n^{(k)}(t), a_c^{(k)}(t))$, $k=1, 2, \ldots$, $t\epsilon[-T, 0]$; given that the defender has k weapons and there has been no attack prior to t, the attacker attacks at t with conditional density $a_n^{(k)}(t)$ if he is type n and $a_c^{(k)}(t)$ if he is type c. That is, $a_n^{(k)}(t)\ \Delta t+o(\Delta t)$ is the probability of an attack in $(t, t+\Delta t)$, given a type n attacker, the defender has k weapons and there has been no attack prior to t. The attack is at $t=0$ if there has been no prior attack.

Let us refer to the state (k, t) when the defender has k weapons at time t. Figure 1 is a generalized game tree which determines the transition from state (k, t) to its possible successors at time $t+\Delta t$; either the game has terminated with one of the payoffs $p_k^{(n)}$, $q_k^{(n)}$, $p_k^{(c)}$, $q_k^{(c)}$ in case of an attack, or is in state $(k, t+\Delta t)$ or $(k-1, t+\Delta t)$ otherwise.

Let

$P_{k,n}(t)=$ defender's survival probability, given k weapons at time t and the future attack is of type n.
$P_{k,c}(t)=$ defender's survival probability, given k weapons at time t and the future attack is of type c.
$P_k(t)=$ defender's survival probability, given k weapons at time t.

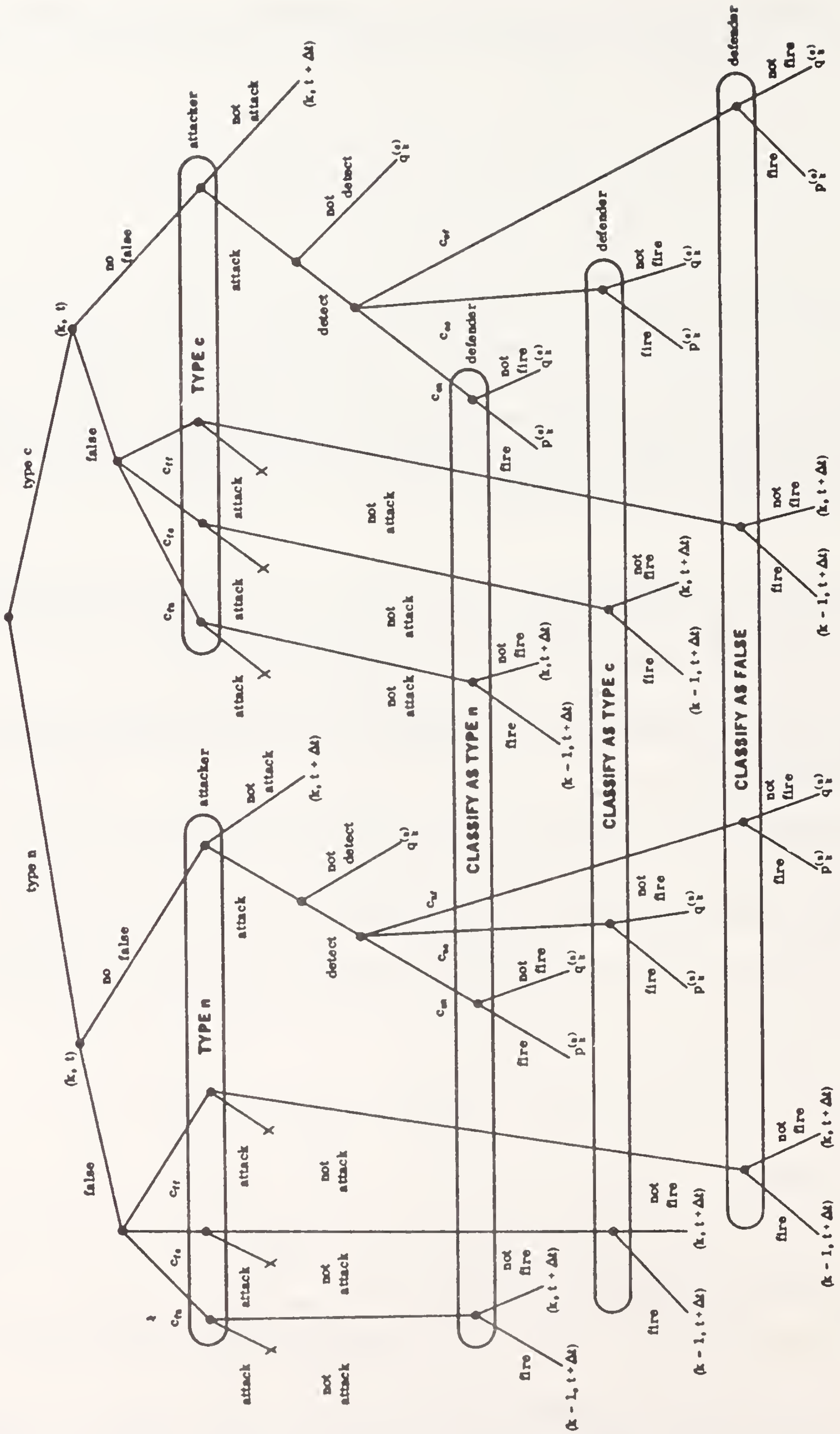
These probabilities are related through straightforward probability calculations (see Fig. 1 for the events which can occur):

$$P_{k,n}(t)=\lambda\Delta t[c_{fn}(1-x_n^{(k)})+c_{fc}(1-x_c^{(k)})+c_{ff}(1-x_f^{(k)})]P_{k,n}(t+\Delta t)+\lambda\Delta t[c_{fn}x_n^{(k)}+c_{fc}x_c^{(k)}$$
$$+c_{ff}x_f^{(k)}]P_{k-1,n}(t+\Delta t)+(1-\lambda\Delta t)(1-a_n^{(k)}(t)\Delta t)P_{k,n}(t+\Delta t)+(1-\lambda\Delta t)a_n^{(k)}(t)\Delta t\{(1-D_n)q_k^{(n)}$$
$$+D_n[c_{nn}(x_n^{(k)}p_k^{(n)}+(1-x_n^{(k)})q_k^{(n)})+c_{nc}(x_c^{(k)}p_k^{(n)}+(1-x_c^{(k)})q_k^{(n)})+c_{nf}(x_f^{(k)}p_k^{(n)}+(1-x_f^{(k)})q_k^{(n)})]\}+o(\Delta t).$$

$$P_{k,c}(t)=\lambda\Delta t[c_{fn}(1-x_n^{(k)})+c_{fc}(1-x_c^{(k)})+c_{ff}(1-x_f^{(k)})]P_{k,c}(t+\Delta t)+\lambda\Delta t[c_{fn}x_n^{(k)}+c_{fc}x_c^{(k)}$$
$$+c_{ff}x_f^{(k)}]P_{k-1,c}(t+\Delta t)+(1-\lambda\Delta t)(1-a_c^{(k)}(t)\Delta t)P_{k,c}(t+\Delta t)+(1-\lambda\Delta t)a_c^{(k)}(t)\Delta t\{(1-D_c)q_k^{(c)}$$
$$+D_c[c_{cn}(x_n^{(k)}p_k^{(c)}+(1-x_n^{(k)})q_k^{(c)}+c_{cc}(x_c^{(k)}p_k^{(c)}+(1-x_c^{(k)})q_k^{(c)})+c_{cf}(x_f^{(k)}p_k^{(c)}+(1-x_f^{(k)})q_k^{(c)})]\}+o(\Delta t).$$

We conclude that the derivatives with respect to t are

(1) $$\dot{P}_{k,n}(t)=a_n^{(k)}(t)\{P_{k,n}(t)-D_n(p_k^{(n)}-q_k^{(n)})(c_{nn}x_n^{(k)}+c_{nc}x_c^{(k)}+c_{nf}x_f^{(k)})-q_k^{(n)}\}+\lambda(c_{fn}x_n^{(k)}$$
$$+c_{fc}x_c^{(k)}+c_{ff}x_f^{(k)})[P_{k,n}(t)-P_{k-1,n}(t)]; \qquad P_{k,n}(0)=D_np_k^{(n)}+(1-D_n)q_k^{(n)}$$


type c
type n
(k, t)
no false
false
attacker
defender
attack
not attack
detect
not detect
fire
not fire
TYPE c
TYPE n
CLASSIFY AS TYPE n
CLASSIFY AS TYPE c
CLASSIFY AS FALSE
(k, t + Δt)
(k - 1, t + Δt)

$$\dot{P}_{k,c}(t)=a_c^{(k)}(t)\{P_{k,c}(t)-D_c(p_k^{(c)}-q_k^{(c)})(c_{cn}x_n^{(k)}+c_{cc}x_c^{(k)}+c_{cf}x_f^{(k)})-q_k^{(c)}]\}+\lambda(c_{fn}x_n^{(k)} \tag{2}$$

$$+c_{fc}x_c^{(k)}+c_{ff}x_f^{(k)})[P_{k,c}(t)-P_{k-1,c}(t)]; \qquad P_{k,c}(0)=D_cp_k^{(c)}+(1-D_c)q_k^{(c)}$$

$$P_k(t)=P(n)P_{k,n}(t)+P(c)P_{k,c}(t).$$

$$\dot{P}_k(t)=x_n^{(k)}\{\lambda c_{fn}[P_k(t)-P_{k-1}(t)]-D_n(p_k^{(n)}-q_k^{(n)})c_{nn}P(n)a_n^{(k)}(t)-D_c(p_k^{(c)}-q_k^{(c)})c_{cn}P(c)a_c^{(k)}(t)\} \tag{3}$$

$$+x_c^{(k)}\{\lambda c_{fc}[P_k(t)-P_{k-1}(t)]-D_n(p_k^{(n)}-q_k^{(n)})c_{nc}P(n)a_n^{(k)}(t)-D_c(p_k^{(c)}-q_k^{(c)})c_{cc}P(c)a_c^{(k)}(t)\}$$

$$+x_f^{(k)}\{\lambda c_{ff}[P_k(t)-P_{k-1}(t)]-D_n(p_k^{(n)}-q_k^{(n)})c_{nf}P(n)a_n^{(k)}(t)-D_c(p_k^{(c)}-q_k^{(c)})c_{cf}P(c)a_c^{(k)}(t)\}$$

$$+P(n)a_n^{(k)}(t)[P_{k,n}(t)-q_k^{(n)}]+P(c)a_c^{(k)}(t)[P_{k,c}(t)-q_k^{(c)}]; \qquad P_k(0)=P(n)P_{k,n}(0)+P(c)P_{k,c}(0).$$

THE SOLUTION

The defender, knowing the terminal condition $P_k(0)=P(n)P_{k,n}(0)+P(c)P_{k,c}(0)$ and wishing $P_k(t)$ to be large, will select his strategy to minimize $\dot{P}_k(t)$. The attacker attempts to make $\dot{P}_k(t)$ large. He accomplishes this by making $\dot{P}_{k,n}(t)$ large if he is type n or $\dot{P}_{k,c}(t)$ large if he is type c.

By definition of the solution of a game, optimal strategies have the property that an optimal strategy is the best response to an optimal strategy, i.e., optimal strategies can be announced in advance without causing either player to change his strategy. Consequently, if an optimal strategy exists for the defender, it is constrained by the conditions that the coefficient of $a_n^{(k)}(t)$ in (1) and the coefficient of $a_c^{(k)}(t)$ in (2) must be less than or equal to zero. If, for example, the coefficient of $a_n^{(k)}$ were strictly positive, then, the larger the value of $a_n^{(k)}$ the better for the attacker. But then $x_n^{(k)}$, $x_c^{(k)}$, and $x_f^{(k)}$ should be one since their coefficients in (3) would be negative, and in turn $a_n^{(k)}$ should be zero since its coefficient would be strictly negative (since $P_{k,n}(t)$ is strictly increasing and has terminal condition $P_{k,n}(0)=D_np_k^{(n)}+(1-D_n)q_k^{(n)}$). With these constraints the best response by the attacker has both $a_n^{(k)}$ times its coefficient and $a_c^{(k)}$ times its coefficient equal to zero.

Hence, if an optimal strategy exists for the defender it must be a solution of the following linear programming (minimization) problem:

Minimize

$$\dot{P}_k(t)=\lambda[c_{fn}x_n^{(k)}+c_{fc}x_c^{(k)}+c_{ff}x_f^{(k)}][P_k(t)-P_{k-1}(t)] \tag{4}$$

subject to constraints

$$\begin{pmatrix} c_{cn} & c_{cc} & c_{cf} \\ c_{nn} & c_{nc} & c_{nf} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_n^{(k)} \\ x_c^{(k)} \\ x_f^{(k)} \end{pmatrix} \geq \begin{pmatrix} a \\ b \\ -1 \\ -1 \\ -1 \end{pmatrix} \tag{5}$$

where

$$a \equiv \frac{P_{k,c}(t) - q_k^{(c)}}{D_c(p_k^{(c)} - q_k^{(c)})}, \qquad b \equiv \frac{P_{k,n}(t) - q_k^{(n)}}{D_n(p_k^{(n)} - q_k^{(n)})}.$$

Observe that a and b are both 1 at $t=0$, otherwise they are strictly in $(0, 1)$.

In order to verify that particular strategies are optimal for the defender and the attacker it suffices to prove that each is a best response to the other. This would seem to involve knowing the coefficients in (1), (2), and (3), or at least knowing whether they are zero, positive, or negative. These coefficients, in turn, are jointly controlled by the players. The problem appears difficult. Luckily, another approach works.

But first, let us solve the minimization problem (4) and (5). We use properties described in [1, chapter 5] and [2, appendix 5].

Introduce slack variables so that the inequalities in (5) become equalities

$$\begin{aligned}
c_{cn}x_n^{(k)} + c_{cc}x_c^{(k)} + c_{cf}x_f^{(k)} - \lambda_1 \qquad\qquad\qquad\qquad &= a\\
c_{nn}x_n^{(k)} + c_{nc}x_c^{(k)} + c_{nf}x_f^{(k)} \qquad - \lambda_2 \qquad\qquad &= b\\
-x_n^{(k)} \qquad\qquad\qquad\qquad - \lambda_3 \qquad &= -1\\
-x_c^{(k)} \qquad\qquad\qquad\qquad - \lambda_4 &= -1\\
-x_f^{(k)} \qquad\qquad\qquad - \lambda_5 &= -1.
\end{aligned}$$

The problem has a solution, since a feasible solution exists ($x_n^{(k)} = x_c^{(k)} = x_f^k = 1$, $\lambda_1 = 1-a$, $\lambda_2 = 1-b$, $\lambda_3 = \lambda_4 = \lambda_5 = 0$). Since there are five equations in eight unknowns there are three nonbasic (zero) variables in the solution. These three cannot be the x's because of the first equation above (note $a > 0$). Neither can all three x's be in the open interval $(0, 1)$, because then $\lambda_3 > 0$, $\lambda_4 > 0$, $\lambda_5 > 0$, leaving only λ_1 and λ_2 as zero variables, a contradiction. Since $a \epsilon (0, 1)$ and $b \epsilon (0, 1)$, the optimal solution has either λ_1 or λ_2 or both equal to zero. For, suppose both were strictly positive. Since at least one of the x's is strictly positive, a reduction in the value of the objective function (4) is possible by decreasing one of the strictly positive x's. We conclude that the optimal solution has one of the forms of Table 1.

Consider the linear programming (maximization) problem which is the dual of (4) and (5):

Maximize

$$aP(c)D_c(p_k^{(c)} - q_k^{(c)})a_c^{(k)} + bP(n)D_n(p_k^{(n)} - q_k^{(n)})a_n^{(k)} - (y_3 + y_4 + y_5) \tag{6}$$

subject to constraints

$$\begin{pmatrix} c_{cn} & c_{nn} & -1 & 0 & 0\\ c_{cc} & c_{nc} & 0 & -1 & 0\\ c_{cf} & c_{nf} & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} P(c)D_c(p_k^{(c)} - q_k^{(c)})a_c^{(k)}\\ P(n)D_n(p_k^{(n)} - q_k^{(n)})a_n^{(k)}\\ y_3\\ y_4\\ y_5 \end{pmatrix} \leqq \begin{pmatrix} \lambda c_{fn}(P_k(t) - P_{k-1}(t))\\ \lambda c_{fc}(P_k(t) - P_{k-1}(t))\\ \lambda c_{ff}(P_k(t) - P_{k-1}(t)) \end{pmatrix}. \tag{7}$$

TABLE 1

	$x_n^{(k)}$	$x_c^{(k)}$	$x_f^{(k)}$
①	$\dfrac{a - c_{cc}x_c^{(k)} - c_{cf}x_f^{(k)}}{c_{cn}}$	0 or 1	0 or 1
②	0 or 1	$\dfrac{a - c_{cn}x_n^{(k)} - c_{cf}x_f^{(k)}}{c_{cc}}$	0 or 1
③	0 or 1	0 or 1	$\dfrac{a - c_{cn}x_n^{(k)} - c_{cc}x_c^{(k)}}{c_{cf}}$
④	$\dfrac{b - c_{nc}x_c^{(k)} - c_{nf}x_f^{(k)}}{c_{nn}}$	0 or 1	0 or 1
⑤	0 or 1	$\dfrac{b - c_{nn}x_n^{(k)} - c_{nf}x_f^{(k)}}{c_{nc}}$	0 or 1
⑥	0 or 1	0 or 1	$\dfrac{b - c_{nn}x_n^{(k)} - c_{nc}x_c^{(k)}}{c_{nf}}$
⑦	$\dfrac{c_{cc}[b - c_{nf}x_f^{(k)}] - c_{nc}[a - c_{cf}x_f^{(k)}]}{c_{nn}c_{cc} - c_{cn}c_{nc}}$	$\dfrac{c_{nn}[a - c_{cf}x_f^{(k)}] - c_{cn}[b - c_{nf}x_f^{(k)}]}{c_{nn}c_{cc} - c_{cn}c_{nc}}$	0 or 1
⑧	0 or 1	$\dfrac{c_{cf}[b - c_{nn}x_n^{(k)}] - c_{nf}[a - c_{cn}x_n^{(k)}]}{c_{nc}c_{cf} - c_{cc}c_{nf}}$	$\dfrac{c_{nc}[a - c_{cn}x_n^{(k)}] - c_{cc}[b - c_{nn}x_n^{(k)}]}{c_{nc}c_{cf} - c_{cc}c_{nf}}$
⑨	$\dfrac{c_{cf}[b - c_{nc}x_c^{(k)}] - c_{nf}[a - c_{cc}x_c^{(k)}]}{c_{nn}c_{cf} - c_{cn}c_{nf}}$	0 or 1	$\dfrac{c_{nn}[a - c_{cc}x_c^{(k)}] - c_{cn}[b - c_{nc}x_c^{(k)}]}{c_{nn}c_{cf} - c_{cn}c_{nf}}$

These constraints are equivalent to

$$
\begin{aligned}
&-y_3 \leq \text{coefficient of } x_n^{(k)} \text{ in (3)} \\
(8) \qquad &-y_4 \leq \text{coefficient of } x_c^{(k)} \text{ in (3)} \\
&-y_5 \leq \text{coefficient of } x_f^{(k)} \text{ in (3)}
\end{aligned}
$$

Table 2 presents a list of possible solutions for the maximization problem. These correspond to the possible solutions of the minimization problem shown in Table 1: The correspondence is ① $\longleftrightarrow \boxed{1}$, ② $\longleftrightarrow \boxed{2}$, . . . , ⑨ $\longleftrightarrow \boxed{9}$. The correspondence is that induced by the property that optimal strategies may be announced. For example ① $\longleftrightarrow \boxed{1}$; $x_n^{(k)}$ is selected to cause the coefficient of $a_c^{(k)}$ in (2) to be zero, while $x_c^{(k)}$ and $x_f^{(k)}$ are either zero or one, according as their coefficients in (3) are positive or negative (we may not know which case pertains). Typically this would require $x_n^{(k)}$ to be strictly in (0, 1). In order that this be optimal it requires that $a_c^{(k)}$ be selected to force the coefficient of $x_n^{(k)}$ to be zero. Then y_3, y_4, y_5 are selected to be as small as possible, i.e., such that the equalities hold in (8), except that negative values are not allowed. This is equivalent to

$$y_3 = -x_n^{(k)} \text{ times coefficient of } x_n^{(k)} \text{ in (3)}$$

$$y_4 = -x_c^{(k)} \text{ times coefficient of } x_c^{(k)} \text{ in (3)}$$

$$y_5 = -x_f^{(k)} \text{ times coefficient of } x_f^{(k)} \text{ in (3).}$$

Let us take one more example, the correspondence $7 \longleftrightarrow \boxed{7}$. Here $x_n^{(k)}$ and $x_c^{(k)}$ are chosen so as to force the coefficients of $a_n^{(k)}$ and $a_c^{(k)}$ in (1) and (2) to be zero. In turn, $a_n^{(k)}$ and $a_c^{(k)}$ are selected to force the coefficients of $x_n^{(k)}$ and $x_c^{(k)}$ to be zero, and y_3, y_4, y_5 are selected as above. $x_f^{(k)}$ is zero or one according as its coefficient is positive or negative. Again, we may not know which case pertains.

TABLE 2

	$a_c^{(k)} \Big/ \dfrac{\lambda[P_k - P_{k-1}]}{P(c)D_c(p_k^{(c)} - q_k^{(c)})}$	$u_n^{(k)} \Big/ \dfrac{\lambda[P_k - P_{k-1}]}{P(n)D_n(p_k^{(n)} - q_k^{(n)})}$	$y_3/\lambda[P_k - P_{k-1}]$	$y_4/\lambda[P_k - P_{k-1}]$	$y_5/\lambda[P_k - P_{k-1}]$
$\boxed{1}$	c_{fn}/c_{cn}	0	0	$x_c^{(k)}(c_{cc}c_{fn}/c_{cn} - c_{fc})$	$x_f^{(k)}(c_{cf}c_{fn}/c_{cn} - c_{ff})$
$\boxed{2}$	c_{fc}/c_{cc}	0	$x_n^{(k)}(c_{cn}c_{fc}/c_{cc} - c_{fn})$	0	$x_f^{(k)}(c_{cf}c_{fc}/c_{cc} - c_{ff})$
$\boxed{3}$	c_{ff}/c_{cf}	0	$x_n^{(k)}(c_{cn}c_{ff}/c_{cf} - c_{fn})$	$x_c^{(k)}(c_{cc}c_{ff}/c_{cf} - c_{fc})$	0
$\boxed{4}$	0	c_{fn}/c_{nn}	0	$x_c^{(k)}(c_{nc}c_{fn}/c_{nn} - c_{fc})$	$x_f^{(k)}(c_{nf}c_{fn}/c_{nn} - c_{ff})$
$\boxed{5}$	0	c_{fc}/c_{nc}	$x_n^{(k)}(c_{nn}c_{fc}/c_{nc} - c_{fn})$	0	$x_f^{(k)}(c_{nf}c_{fc}/c_{nc} - c_{ff})$
$\boxed{6}$	0	c_{ff}/c_{nf}	$x_n^{(k)}(c_{nn}c_{ff}/c_{nf} - c_{fn})$	$x_c^{(k)}(c_{nc}c_{ff}/c_{nf} - c_{fc})$	0
$\boxed{7}$	$\dfrac{c_{nn}c_{fc} - c_{nc}c_{fn}}{c_{nn}c_{cc} - c_{nc}c_{cn}}$	$\dfrac{c_{fn}c_{cc} - c_{cn}c_{fc}}{c_{nn}c_{cc} - c_{nc}c_{cn}}$	0	0	$x_f^{(k)}\left\{-c_{ff} + c_{cf}\dfrac{c_{nn}c_{fc} - c_{nc}c_{fn}}{c_{nn}c_{cc} - c_{nc}c_{cn}} + c_{nf}\dfrac{c_{fn}c_{cc} - c_{cn}c_{fc}}{c_{nn}c_{cc} - c_{nc}c_{cn}}\right\}$
$\boxed{8}$	$\dfrac{c_{nc}c_{ff} - c_{nf}c_{fc}}{c_{nc}c_{cf} - c_{cc}c_{nf}}$	$\dfrac{c_{fc}c_{cf} - c_{ff}c_{cc}}{c_{nc}c_{cf} - c_{cc}c_{nf}}$	$x_n^{(k)}\left\{c_{nn}\dfrac{c_{fc}c_{cf} - c_{ff}c_{cc}}{c_{nc}c_{cf} - c_{cc}c_{nf}} + c_{cn}\dfrac{c_{nc}c_{ff} - c_{nf}c_{fc}}{c_{nc}c_{cf} - c_{cc}c_{nf}} - c_{fn}\right\}$	0	0
$\boxed{9}$	$\dfrac{c_{nn}c_{ff} - c_{nf}c_{fn}}{c_{nn}c_{cf} - c_{nf}c_{cn}}$	$\dfrac{c_{fn}c_{cf} - c_{cn}c_{ff}}{c_{nn}c_{cf} - c_{nf}c_{cn}}$	0	$x_c^{(k)}\left\{c_{nc}\dfrac{c_{fn}c_{cf} - c_{cn}c_{ff}}{c_{nn}c_{cf} - c_{nf}c_{cn}} + c_{cc}\dfrac{c_{nn}c_{ff} - c_{nf}c_{fn}}{c_{nn}c_{cf} - c_{nf}c_{cn}} - c_{fc}\right\}$	0

We do know that under the correspondence $1 \longleftrightarrow \boxed{1}$, $2 \longleftrightarrow \boxed{2}$, . . ., $9 \longleftrightarrow \boxed{9}$, the objective functions (4) and (6) are identical. Furthermore, among the finite number of forms that are candidates for basic feasible solutions of the maximization problem, only $\boxed{i}$ causes its objective function to be identical with the objective function under i, $i = 1, 2, \ldots, 9$. Any other expression of equality of the objective functions is not an identity. There are finitely many such expressions since there are finitely many basic feasible solutions.

We have spoken above of *forms* that are candidates for basic feasible solutions. We mean those

vectors that are basic feasible solutions for some parameter values. A form amounts to specifying those variables that are basic, along with their values, disregarding the conditions under which they are feasible.

The variables in a form are continuous functions of the parameters. Consider the set of basic feasible solutions of the minimization problem and the maximization problem for fixed parameters. If a form is not feasible then at least one of the constraints is violated strictly. Since a form is continuous, this violation remains for all sufficiently small changes in the parameters. Therefore, if a form is excluded from the set of basic feasible solutions for a particular set of parameter values, then for all sufficiently small changes in the parameters this form remains excluded.

Suppose ⓘ is the solution of the minimization problem for a particular set of parameter values and let $\mathcal{S} = (S_1 S_2, \ldots, S_N)$ be the set of basic feasible solutions of the dual (maximization) problem. For all sufficiently small changes in the parameters, the set $\mathcal{S}$ can add no new forms. Suppose $\boxed{i}$, the form corresponding to ⓘ is not an element of $\mathcal{S}$. Let the objective function for the maximization problem when S_j is used be $L(S_j)$ and the objective function for the minimization problem when ⓘ is used be L_0. There exist small changes in the parameters such that ⓘ remains optimal, $\boxed{i}$ is not added to $\mathcal{S}$ and none of the equalities $L(S_j) = L_0$, $j = 1, 2, \ldots, N$, hold. But this is a contradiction, since the optimal values of the objective functions in dual linear programming problems must be equal. We conclude that $\boxed{i} \in \mathcal{S}$ and is the solution of the maximization problem. We have proved

THEOREM 1: *At time* t, *when the defender has* k *weapons,*

(i) *the value* $V_k(t)$ *is that unique function whose derivative at* t *is the optimal value of* (4) *with terminal condition specified in* (3),

(ii) *the defender has an optimal strategy which is the solution of the L.P. program* (4) *and* (5), *i.e., the strategy in Table 1 which minimizes* (4),

(iii) *the attacker has an optimal strategy which is the solution of the dual L.P. program* (6) *and* (7), *i.e., the strategy* $\boxed{j}$ *in Table 2 which corresponds to the solution* ⓙ *of Table 1.*

COROLLARY: *Let* $f(t) = \int_t^0 \lambda(s)ds$, *the expected number of false targets in the interval* [t, 0]. *Then the value* $V_k(t)$ *depends on* λ *and* t *only through* f(t).

PROOF: Let $V_k^*[-f(t)]$ be the unique solution of the system of differential equations with form (4) and terminal condition $V_k^*(0) = P_k(0)$, where $V_k^*[-f(t)]$ replaces $P_k(t)$, $V_{k-1}^*[-f(t)]$ replace $P_{k-1}(t)$, $\lambda(t)$ is identically one, and $-f(t)$ replaces t. Then

$$\frac{dV_k^*[-f(t)]}{d[-f(t)]} = \frac{dV_k^*[-f(t)]}{dt} \frac{1}{d[-f(t)]/dt}$$

$$= \frac{1}{\lambda(t)} \frac{dV_k^*[-f(t)]}{dt},$$

so that $V_k^*[-f(t)]$ satisfies the same system of differential equations and the same condition as as $V_k(t)$. Hence, $V_k^*[-f(t)] = V_k(t) \ \forall \ (k, t)$.

THEOREM 2: *Let* $p^{(n)}, q^{(n)}, p^{(c)}, q^{(c)}$ *be the limits as* $k \to \infty$ *of* $p_k^{(n)}, q_k^{(n)}, p_k^{(c)}, q_k^{(c)}$, *respectively, and let* $f(t) = \int_t^0 \lambda(s)ds$ = *expected number of false targets in* [t, 0].

(i) $V_k(t)$ *has a limit as* $-t \to \infty$ *with* $k/f(t)$ *constant: This limit is the maximum value of*

$$c_1x_n + c_2x_c + c_3x_f + P(c)q^{(c)} + P(n)q^{(n)} \tag{9}$$

subject to

$$\begin{pmatrix} c_{fn} & c_{fc} & c_{ff} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ x_c \\ x_f \end{pmatrix} \leqslant \begin{pmatrix} \min\,[1, k/f(t)] \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tag{10}$$

where

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_{cn} & c_{nn} \\ c_{cc} & c_{nc} \\ c_{cf} & c_{nf} \end{pmatrix} \begin{pmatrix} P(c)D_c(p^{(c)} - q^{(c)}) \\ P(n)D_n(p^{(n)} - q^{(n)}) \end{pmatrix}. \tag{11}$$

(ii) *For* (x_n, x_c, x_f) *a solution of* (9) *and* (10) *and* $\epsilon > 0$ *arbitrary, let* $(\alpha, \beta, \gamma) = (\max\,(x_n - \epsilon, 0), \max\,(x_c - \epsilon, 0), \max\,(x_f - \epsilon, 0))$ *be the defender's firing probabilities when the classification is type* n, *type* c, *or false target. As* $-t \to \infty$ *with* $k/f(t)$ *constant, the limit inferior of the minimum probability of survival under* (α, β, γ) *is at least as large as the limiting value of* $V_k(t) - \epsilon$.

(iii) *The solution of the L. P. problem* (9) *and* (10) *is one of the forms presented in Table 3.*

TABLE 3

Limiting value	x_n	x_c	x_f
$c_1 \dfrac{\min\,[1, k/f(t)] - c_{fc}x_c - c_{ff}x_f}{c_{fn}} + c_2x_c + c_3x_f + P(c)q^{(c)} + P(n)q^{(n)}$	$\dfrac{\min\,[1, k/f(t)] - c_{fc}x_c - c_{ff}x_f}{c_{fn}}$	0 or 1	0 or 1
$c_1x_n + c_2 \dfrac{\min\,[1, k/f(t)] - c_{fn}x_n - c_{ff}x_f}{c_{fc}} + c_3x_f + P(c)q^{(c)} + P(n)q^{(n)}$	0 or 1	$\dfrac{\min\,[1, k/f(t)] - c_{fn}x_n - c_{ff}x_f}{c_{fc}}$	0 or 1
$c_1x_n + c_2x_2 + c_3 \dfrac{\min\,[1, k/f(t)] - c_{fn}x_n - c_{fc}x_c}{c_{ff}} + P(c)q^{(c)} + P(n)q^{(n)}$	0 or 1	0 or 1	$\dfrac{\min\,[1, k/f(t)] - c_{fn}x_n - c_{fc}x_c}{c_{ff}}$

$c_1 = P(c)D_c(p^{(c)} - q^{(c)})c_{cn} + P(n)D_n(p^{(n)} - q^{(n)})c_{nn}$
$c_2 = P(c)D_c(p^{(c)} - q^{(c)})c_{cc} + P(n)D_n(p^{(n)} - q^{(n)})c_{nc}$
$c_3 = P(c)D_c(p^{(c)} - q^{(c)})c_{cf} + P(n)D_n(p^{(n)} - q^{(n)})c_{nf}$

PROOF: For convenience we prove the theorem for $p^{(n)}=p_1^{(n)}=p_2^{(n)}=\ldots$, $q^{(n)}=q_1^{(n)}=q_2^{(n)}=\ldots$, $p^{(c)}=p_1^{(c)}=p_2^{(c)}=\ldots$, $q^{(c)}={}_1^{(c)}=q_2^{(c)}=\ldots$ and remark that the theorem extends with no difficulty to the more general case.

We first prove (iii).

Introduce slack variables so that equalities hold in the set of constraints (10)

$$\begin{aligned}
c_{fn}x_n + c_{fc}x_c + c_{ff}x_f + \mu_1 &= \min\,[1, k/f(t)] \\
x_n + \mu_2 &= 1 \\
x_c + \mu_3 &= 1 \\
x_f + \mu_4 &= 1.
\end{aligned}$$

Since there are four equations in seven unknowns there are three nonbasic (zero) variables. In the solution it is clear that $\mu_1=0$. Therefore, at least one of the variables x_n, x_c, x_f is strictly positive and hence basic. It cannot be the case that more than one of these variables is in (0, 1). For, suppose, for example, that x_n and x_c were in (0, 1). Then $\mu_2>0$, $\mu_3>0$. This leaves only x_f, μ_1, μ_4 to be zero. Not both x_f and μ_4 can be zero so we obtain a contradiction to the fact of three nonbasic variables. Likewise we obtain a contradiction if x_n, x_c, x_f were all in (0, 1). The proof of (iii) is complete.

Throughout the interval $[t, 0]$ let the defender employ the strategy (α, β, γ) of (ii). Then, the probability of survival (and hence $V_k(t)$) is at least as large as the product of $\min\limits_{s\epsilon[t,0]} Pr$ (defender possesses a weapon at time s).
and

$$\begin{aligned}
&P(n)D_n\{c_{nn}[\alpha p^{(n)}+(1-\alpha)\,q^{(n)}]+c_{nc}\,[\beta p^{(n)}+(1-\beta)\,q^{(n)}]+c_{nf}\,[\gamma p^{(n)}+(1-\gamma)\,q^{(n)}]\} \\
&\quad +P(n)\,(1-D_n)\,q^{(n)}+P(c)D_c\{c_{cn}\,[\alpha p^{(c)}+(1-\alpha)\,q^{(c)}]+c_{cc}\,[\beta p^{(c)}+(1-\beta)\,q^{(c)}]+c_{cf}\,[\gamma p^{(c)} \\
&\quad\quad +(1-\gamma)\,q^{(c)}]\}+P(c)\,(1-D_c)\,q^{(c)}\equiv c_1\alpha+c_2\beta+c_3\gamma+P(c)\,q^{(c)}+P(n)\,q^{(c)} \\
&\quad\quad\quad \geqslant c_1x_n+c_2x_c+c_3x_f+P(c)\,q^{(c)}+P(n)\,q^{(n)}-\epsilon.
\end{aligned}$$

The strategy (α, β, γ) was chosen so that the first of these factors tends to one as $-t\to\infty$ with $k/f(t)$ constant. Hence,

$$\liminf_{\substack{-t\to\infty \\ k/f(t)\text{ constant}}} V_k(t) \geqslant c_1x_n+c_2x_c+c_3x_f+P(c)q^{(c)}+P(n)q^{(n)}-\epsilon.$$

Given state (k, t), suppose the attacker selects the attack time $s\,\epsilon\,[t, 0]$ with density $\lambda(s)/f(t)$; $f(t)=\int_t^0 \lambda(s)ds$. Then, given F false targets in $[t, 0]$, all arrangements $(A, f_1, f_2, \ldots, f_F)$, $(f_1, A, f_2, \ldots, f_F)$, $(f_1, f_2, \ldots, f_F, A)$ of the position of attack among the false targets are equally likely, i.e., each has probability $\frac{1}{1+F}\cdot$ Among the F false targets N_{fn}, N_{fc}, N_{ff} are the numbers of false targets classified as type n, type c, or false, respectively. Let $N_{fn}\alpha_n$, $N_{fc}\alpha_c$, $N_{ff}\alpha_f$ be the number of weapons

fired at classifications of type n, type c, or false target, respectively. If the attacker is classified as type n then there are $N_{fn}+1$ arrangements of the position of the attacker among the N_{fn} false targets classified as type n. Similar statements hold if the attacker is classified as type c or as a false target. Then, given N_{fn}, N_{fc}, N_{ff} the probability of survival is

(12) $$\begin{aligned} Pr(\text{survival}\,|\,N_{fn}, N_{fc}, N_{ff}) = P(c)D_c \Big\{ & c_{cn}\left[\frac{N_{fn}\alpha_n}{N_{fn}+1}\,p^{(c)} + \left(1-\frac{N_{fn}\alpha_n}{N_{fn}+1}\right)q^{(c)}\right] + c_{cc}\left[\frac{N_{fc}\alpha_c}{N_{fc}+1}\,p^{(c)} \right. \\ & \left. + \left(1-\frac{N_{fc}\alpha_c}{N_{fc}+1}\right)q^{(c)}\right] + c_{cf}\left[\frac{N_{ff}\alpha_f}{N_{ff}+1}\,p^{(c)} + \left(1-\frac{N_{ff}\alpha_f}{N_{ff}+1}\right)q^{(c)}\right]\Big\} \\ & + P(c)(1-D_c)q^{(c)} + P(n)D_n\Big\{c_{nn}\left[\frac{N_{fn}\alpha_n}{N_{fn}+1}\,p^{(n)} + \left(1-\frac{N_{fn}\alpha_n}{N_{fn}+1}\right)q^{(n)}\right] \\ & + c_{nc}\left[\frac{N_{fc}\alpha_c}{N_{fc}+1}\,p^{(n)} + \left(1-\frac{N_{fc}\alpha_c}{N_{fc}+1}\right)q^{(n)}\right] \\ & + c_{nf}\left[\frac{N_{ff}\alpha_f}{N_{ff}+1}\,p^{(n)} + \left(1-\frac{N_{ff}\alpha_f}{N_{ff}+1}\right)q^{(n)}\right]\Big\} + P(n)(1-D_n)q^{(n)} \\ = {} & c_1\frac{N_{fn}}{N_{fn}+1}\,\alpha_n + c_2\frac{N_{fc}}{N_{fc}+1}\,\alpha_c + c_3\frac{N_{ff}}{N_{ff}+1}\,\alpha_f + P(c)q^{(c)} + P(n)q^{(n)}, \end{aligned}$$

where c_1, c_2, and c_3 are defined by (11).

Consider the L. P. problem: Select nonnegative α_n, α_c, α_f so as to maximize $Pr(\text{survival}\,|\,N_{fn}, N_{fc}, N_{ff})$

subject to

(13) $$\begin{pmatrix} N_{fn} & N_{fc} & N_{ff} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_n \\ \alpha_c \\ \alpha_f \end{pmatrix} \leqslant \begin{pmatrix} \min\,[F, k] \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Dividing the first inequality by $f(t)$ and letting $-t$ tend to infinity with $k/f(t)$ constant, we have that with probability one this L. P. problem tends to the L. P. problem (9) and (10). Since

$$Pr(\text{survival}) \leqslant \sum_{N_{fn}, N_{fc}, N_{ff}} Pr(N_{fn}, N_{fc}, N_{ff}) \max Pr(\text{survival}\,|\,N_{fn}, N_{fc}, N_{ff})$$

we have $V_k(t)$ is bounded above by the right hand member of this inequality which, with probability one, tends to the maximum value of (9). Hence,

$$\limsup_{\substack{-t\to\infty \\ k/f(t)\ \text{constant}}} V_k(t) \leqslant \text{maximum value of (9)}.$$

Combining with a previous result we have

$$\limsup_{\substack{t\to\infty \\ k/f(t)\text{ constant}}} V_k(t) \leqslant \liminf_{\substack{t\to\infty \\ k/f(t)\text{ constant}}} V_k(t) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, (i) and (ii) follow.

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ON THE DISCRETE-TIME QUEUE LENGTH DISTRIBUTION UNDER MARKOV-DEPENDENT PHASES

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ABSTRACT

The technique of probability generating functions has been applied to solve the steady state behavior of a discrete-time, single-channel, queueing problem wherein the arrivals to the queue at consecutive time-marks are statistically independent, but the service is accomplished in phases which are Markov-dependent. Special cases of importance have been discussed. In the end, mean number of phases, its special cases, the mean queue lengths, and the variances have been ascertained.

INTRODUCTION

Many queuing problems are solved on the assumption that the arrivals to the queue or departures from the queue during two successive time-periods are statistically independent. Chaudhry [3–5] investigates a certain class of queuing problems in which the assumption of independence is relaxed in the sense that either the arrivals or the departures are correlated. Other types of dependence have also been discussed in the literature, e.g., Runnenburg [10, 11] discusses dependence of interarrival times. For various other dependences one may see a recent study by Bhat [1]. The present study in a sense is an extension of the author's paper [3]. The stochastic transition probability matrix considered in this study is unsymmetrical and more realistic in nature than the doubly stochastic matrix he discussed in [3, 4]; the former case having been derived as a particular case of the present study. Besides, the two models considered in Ref. [3] have been condensed into one.

In this paper we study the steady state behavior of a discrete-time, single-channel, first-come-first-served queuing problem wherein the service-phases at two consecutive epochs of time (defined later) are Markov-dependent. An account of the type of phases which we are going to consider here is given by Saaty [12]. Following an approach similar to that of Chaudhry [3, 4], we have derived steady-state probability generating functions for the number of phases waiting and in service and stated their expression under special conditions.

Many queuing problems involve hyman beings whose influence, therefore, on the causes and remedies of queuing problems cannot be completely divorced. Suppose, for example, an anti-aircraft gunner is shooting down airplanes. Thus, if we identify a departure (or no departure) from the queue with the shooting down (or not shooting down) of an airplane by the anti-aircraft gunner, then

1. If an airplane was shot down at an epoch of time, then
   (a) Prob. (next airplane will be shot down at the next epoch of time) $= p_1$
   (b) Prob. (next airplane will not be shot down at the next epoch of time) $= 1-p_1$
2. If an airplane was *not* shot down at an epoch of time, then
   (a) Prob. (next airplane will be shot down at the next epoch of time) $= 1-p_2$
   (b) Prob. (next airplane will not be shot down at the next epoch of time) $= p_2$

Assume that the time is divided into denumerable set of epochs of time $t_0 < t_1 < t_2 \ldots\ldots\ldots$, and let the various events involved in the process occur only at these epochs. It is further assumed that the probability of more than one arrival or of completion of more than one phase of a unit admitted into service at a particular epoch is neglected.

Consider two consecutive epochs t_{r-1}, t_r, $r=1, 2, \ldots$, and let the probabilities of a completion or noncompletion of a phase at t_r be governed by the unsymmetrical transition probability matrix (*t.p.m.*).

$$\begin{array}{lcc} & \text{Completion of a phase at } t_r & \text{Noncompletion of a phase at } t_r \\ \text{Completion of a phase at } t_{r-1} & \left\| \begin{array}{c} p_1 \\ 1-p_2 \end{array} \right. & \left. \begin{array}{c} 1-p_1 \\ p_2 \end{array} \right\| \\ \text{Noncompletion of a phase at } t_{r-1} & & \end{array}$$

Hence there is a correlation between phases at two consecutive epochs. In passing, it may be mentioned here that the correlation exists only if there is at least one phase in the system (queue+ service). This is because of the simple fact that when the system is empty, the phases cannot be correlated. Let us then assume that if the system is empty at $t_r - 0$ and a unit arrives at t_r,

Prob. (the unit completes its phase at t_r) $= \alpha$

Prob. (the unit does not complete its phase at t_r) $= 1-\alpha$, and further that if the system is empty at $t_r - 0$ and no arrival occurs at t_r,

Prob. (the phase is completed) $= 0$.

Prob. (the phase is not completed) $= 1$.

Let the probability for an arrival be λ and consequently for no arrival $\mu = (1-\lambda)$. Let C_j be the probability that an arrival would demand that all the j phases of his service be completed before another unit was admitted into service. It may be remarked here that Luchak [8] and Gaver [6] have observed that a wide class of service-time distributions can be obtained or approximated to by varying C_j.

PROBABILITY GENERATING FUNCTION FOR THE NUMBER OF PHASES WAITING AND IN SERVICE IN THE STEADY STATE

Define:

$P_n =$ Prob. (queue is in state n* at $t_{r+1} - 0$, a phase changes at t_r)

$Q_n =$ Prob. (queue is in state n at $t_{r+1} - 0$, phase does not change at t_r).

Elementary probability reasoning gives the following mutually exclusive and exhaustive ways through which P_n is determined.

(i) There are $(n+1)$ phases at $t_r - 0$, a phase was completed at t_{r-1}, and at t_r no arrival occurs, but a phase is completed.

(ii) There are $(n+1)$ phases at $t_r - 0$, a phase was *not* completed at t_{r-1}, and at t_r no arrival occurs, but a phase is completed.

* By state n we mean that there are n phases in the system, including those in service, if any.

(iii) There are $(n-j+1)$ phases at $t_r - 0$, a phase was completed at t_{r-1}, and at t_r an arrival occurs; demands for j phases $(j=1, 2, \ldots, n)$ and a phase is completed.

(iv) There are $(n-j+1)$ phases at $t_r - 0$, a phase was *not* completed at t_{r-1}, and at t_r an arrival occurs; demands for j phases $(j=1, 2, \ldots, n)$ and a phase is completed.

(v) There is no phase in the system at $t_r - 0$, a phase was completed at t_{r-1}, and at t_r an arrival occurs; demands for $(n+1)$ phases and a phase is completed.

(vi) There is no phase in the system at $t_r - 0$, a phase was *not* completed at t_{r-1}, and at t_r an arrival occurs; demands for $(n+1)$ phases and a phase is completed.

Thus

$$P_n = \mu[p_1 P_{n+1} + (1-p_2)Q_{n+1}] + \lambda\left[p_1 \sum_{j=1}^{n} C_j P_{n-j+1} + (1-p_2)\sum_{j=1}^{n} C_j Q_{n-j+1}\right] + \lambda\alpha C_{n+1}(P_0 + Q_0) \qquad n > 0. \tag{1}$$

By similar reasoning

$$P_0 = \mu[p_1 P_1 + (1-p_2)Q_1] + \lambda\alpha C_1(P_0 + Q_0). \tag{2}$$

It is to be noted that $P_j = 0$ for $j < 0$.

Arguing as above we have the following recurrence relations for Q_n's:

$$Q_n = \mu[(1-p_1)P_n + p_2 Q_n] + \lambda\left[(1-p_1)\sum_{j=1}^{n-1} C_j P_{n-j} + p_2 \sum_{j=1}^{n-1} C_j Q_{n-j}\right] + \lambda C_n(P_0 + Q_0)(1-\alpha) \qquad n > 1, \tag{3}$$

$$Q_1 = \mu[(1-p_1)P_1 + p_2 Q_1] + \lambda C_1(1-\alpha)(P_0 + Q_0), \tag{4}$$

$$Q_0 = \mu(P_0 + Q_0), \tag{5}$$

and

$$Q_j = 0 \qquad \text{for } j < 0.$$

Let us introduce the generating functions

$$P(\theta) = \sum_{n=0}^{\infty} \theta^n P_n \qquad |\theta| \leq 1 \tag{6}$$

and

$$Q(\theta) = \sum_{n=0}^{\infty} \theta^n Q_n.$$

Multiplying (1) by θ^{n+1}, summing over all n, using (2) after multiplying it by θ, we have on rearrangement and after using (6)

(7) $$P(\theta)[p_1(\mu+\lambda C(\theta))-\theta]+Q(\theta)[(1-p_2)(\mu+\lambda C(\theta)]+k=0,$$

where

$$C(\theta)=\sum_{j=1}^{\infty} C_j\theta^j,$$

$$k=\frac{Q_0}{\mu}[\lambda\alpha C(\theta)-(\mu+\lambda C(\theta))(\lambda p_1+\mu(1-p_2))],$$

$$\text{and} \sum_{i=0}^{\infty}\sum_{m=0}^{i} a_m b_i=\sum_{m=0}^{\infty}\sum_{i-m=0}^{\infty} a_m b_i.$$

Similarly from (3), (4), (5), and (6), we have

(8) $$P(\theta)[(1-p_1)(\mu+\lambda C(\theta))]+Q(\theta)[p_2(\mu+\lambda C(\theta))-1]-k=0.$$

Add (7) and (8)

(9) $$P(\theta)[\mu+\lambda C(\theta)-\theta]+Q(\theta)[\lambda(C(\theta)-1)]=0.$$

Solve (8) and (9)

$$P(\theta)=\frac{N_1}{D} \quad \text{and } Q(\theta)=\frac{N_2}{D},$$

where

$$N_1=\lambda k(C(\theta)-1),$$

$$N_2=-k[\mu+\lambda C(\theta)-\theta],$$

and

$$D=[\mu+\lambda C(\theta)][\lambda p_1+\mu(1-p_2)+\lambda C(\theta)(1-p_1-p_2)+p_2\theta]-\theta,$$

whence

(10) $$R(\theta)\equiv P(\theta)+Q(\theta)=\frac{N_1+N_2}{D}=\frac{N}{D},$$

where $N=k(\theta-1)$.

Now when $\theta\to 1$, $C(1)=1$ and therefore (10) takes an indeterminate form. Differentiating the numerator and the denominator of the expression given in (10) and then setting $\theta=1$, we get

(10a) $$\frac{Q_0}{\mu}[\lambda\alpha-(\lambda p_1+\mu(1-p_2))]=\lambda L_C(2-p_1-p_2)+(p_2-1),$$

where

$$L_C=C'(1)=\sum_{j=1}^{\infty} jC_j.$$

Finally

$$R(\theta)=\frac{(\theta-1)[\lambda L_C(2-p_1-p_2)+(p_2-1)][\lambda\alpha C(\theta)-\{\mu+\lambda C(\theta)\}\{\lambda p_1+\mu(1-p_2)\}]}{[\lambda\alpha-\{\lambda p_1+\mu(1-p_2)\}][\{\mu+\lambda C(\theta)\}\{\lambda p_1+\mu(1-p_2)+\lambda C(\theta)(1-p_1-p_2)+p_2\theta\}-\theta]}.$$
(11)

NECESSARY CONDITIONS FOR THE EXISTENCE OF STEADY-STATE

The conditions for the existence of steady-state may be discussed by two different procedures.

Firstly, the steady-state exists if $R_0>0$. In an uncorrelated study, this type of argument has been used earlier by Hawkes [7]. Now by using Equations (10a) and (5), $R_0>0$ implies

(a) $$\lambda L_C<(1-p_2)/(2-p_1-p_2)$$

and

(b) $$\lambda\alpha<\lambda p_1+\mu(1-p_2),$$

the latter inequality being redundant. For whenever the inequality (a) is satisfied, the inequality (b) is also satisfied. The proof of this runs on the following lines. The right side of the inequality (a) being a probability (this is shown in the alternative proof for the existence of steady-state) is, in general, a fraction, and consequently it is clear that $\lambda <$ a certain fraction. But λ, p_1, μ, $1-p_2$ being probabilities, it is trivial to show that $\lambda p_1+\mu(1-p_2)<2$. It now follows from the inequality (b) that $\lambda<2/\alpha$ and as α is also a probability lying between 0 and 1, $\lambda <$ a quantity which $\geqslant 2$. From the above argument, it is clear that the necessary condition for the existence of steady state is only

$$\lambda L_C<(1-p_2)/(2-p_1-p_2),$$

which is equivalent to the usual restriction on the traffic intensity in the classical steady-state queuing systems.

It may be remarked here that $R_0>0$ even if the inequalities in (a) and (b) are reversed. But this is not true. For, then, (b) would imply $\lambda>2/\alpha$ which is impossible.

Alternatively, we can derive the above restriction as follows. We know from the theory of stochastic matrices (see, e.g., Parzen [9]) that in the long-run,

P(phase is completed) $=(1-p_2)/(2-p_1-p_2)$

P(phase is not completed) $=(1-p_1)/(2-p_1-p_2)$

provided $0<p_1, p_2<1$.

Now λL_C being the expected number of phases created by an arriving unit $< (1-p_2)/(2-p_1-p_2)$ by the usual argument that the expected input should be less than the expected out under the steady-state conditions.

Having established the steady-state conditions, we now proceed to show that the generating function given in (11) can be varied by imposing different conditions which we proceed to state:

(i) Should the service-times be Pearson Type III (i.e., $C_j=\delta_{jk}$ is the Kronecker symbol), $C(\theta)=\theta^k$, $L_C=k$.

(ii) Should we make the substitutions

$$p_1=p_2=p$$

$$1-p_1=1-p_2=q,\ p+q=1,^*$$

i.e., should the correlation between the service phases be given by a doubly stochastic t.p.m.

(iii) Should $\alpha=0$. This case may be interpreted that if the system is empty at t_r-0, then the phase of a unit arriving at t_r cannot change at t_r.

(iv) Should $\alpha=1/2$. This case may be interpreted that if the system is empty at t_r-0, and if a unit arrives at t_r, then

Prob. (phase of the unit changes at t_r) $=1/2$,

Prob. (phase of the unit does not change at t_r) $=1/2$. The value 1/2 has been taken here because it seems natural to have probability for a change or no change of a phase as half when the queue is in state zero and an arrival occurs.

(v) Should $r=1$, i.e., when there is a perfect correlation between the service phases, $p=1$, $q=0$.

(vi) Should $r=-1$, i.e., when there is a perfect negative correlation between the phases, $p=0$, $q=1$.

(vii) Should $r=0$, i.e., when the phases are uncorrelated, $p=q=1/2$.

(viii) Should $C_j=\delta_{j1}$, i.e., when the number of phases reduce to one, $C(\theta)=\theta$, $L_C=1$, and then P_n, Q_n become the respective steady state probabilities that there are n units waiting and in service at t_r-0 according as a *departure* has or has not occurred at t_{r-1}.

We can find the expression for $R(\theta)$ given in (11) under various combinations of the conditions stated in (i) through (viii). Here we take a few of them:

(a) If conditions given in (ii) and (iii) are satisfied,

$$R(\theta)=\frac{q(\theta-1)(2\lambda L_C-1)(\mu+\lambda C(\theta))}{\{\mu+\lambda C(\theta)\}\{\lambda p+\mu q+\lambda C(\theta)(q-p)+p\theta\}-\theta}.$$

(b) If conditions given in (ii), (iii), and (viii) are satisfied,

$$R(\theta)=\frac{q(\lambda-\mu)(\mu+\lambda\theta)}{\lambda(\lambda q+\mu p)\theta-\mu(\lambda p+\mu q)}. \tag{12}$$

*The coefficient of correlation between the service phases in this case has been ascertained to be $p-q=r$, say, by Chaudhry [3], only when there is at least one phase in the system.

The result in Equation (12) agrees with the one in Equation (12), Chaudhry [3]. It may be noted that this result is true under the conditions that if the system is empty at t_r-0 and if an arrival occurs at t_r, then

Prob. (the arrival departs at t_r) $=0$,

Prob. (the arrival does not depart at t_r) $=1$.

The probability R_n may be picked up as coefficient of θ^n in the binomial expansion of $R(\theta)$ in Equation (12) and is given by

$$R_n=\frac{(\mu-\lambda)q}{(\lambda p+\mu q)(\lambda q+\mu p)}\cdot\left[\frac{\lambda}{\mu}\,\frac{\lambda q+\mu p}{\lambda p+\mu q}\right]^n, \qquad n\geqslant 1$$

which is a geometric distribution.

If further the departures are uncorrelated, i.e., when $p=q=1/2$, then

$$R_n=2(\mu-\lambda)\left(\frac{\lambda}{\mu}\right)^n, \qquad n\geqslant 1$$

a result discussed by Breiman [2].

(c) If the conditions given in (ii) and (iv) are satisfied,

$$R(\theta)=\frac{q(\theta-1)(2\lambda L_C-1)[\lambda C(\theta)-2\{\mu+\lambda C(\theta)\}(\lambda p+\mu q)]}{[\lambda-2(\lambda p+\mu q)][\{\mu+\lambda C(\theta)\}\{\lambda p+\mu q+\lambda C(\theta)(q-p)+p\theta\}-\theta]}.$$

(d) If the conditions given in (ii), (iv), and (viii) are satisfied,

$$R(\theta)=\frac{q(\lambda-\mu)[\lambda\theta-2(\mu+\lambda\theta)(\lambda p+\mu q)]}{[\lambda-2(\lambda p+\mu q)][\lambda(\lambda q+\mu p)\theta-\mu(\lambda p+\mu q)]}. \tag{13}$$

The result in Equation (13) is in conformity with the one in Equation (33), Chaudhry [3]. This result is true under the conditions that if the system is empty at t_r-0 and if an arrival occurs at t_r, then 1/2 is the probability that the arrival will depart at t_r, and 1/2 is the probability that the arrival will not depart at t_r.

The explicit probability in this case is

$$R_n=\frac{q(\mu-\lambda)}{(\lambda q+\mu p)\quad(2\lambda p+\mu q)-\lambda}\cdot\left[\frac{\lambda}{\mu}\,\frac{\lambda q+\mu p}{\lambda p+\mu q}\right]^n, \qquad n\geqq 1,$$

which is once again a geometric distribution.

(e) If the conditions given in (ii), (iii), and (vii) are satisfied,

$$R(\theta)=\frac{(\theta-1)(2\lambda L_C-1)(\mu+\lambda C(\theta))}{(\theta+1)(\mu+\lambda C(\theta))-2\theta}. \tag{14}$$

The result in Equation (14) tallies with the one given in line 7, p. 471, Chaudhry [4], if we first inter-

change p, q to λ, μ, respectively, and then put $p=q=1/2$ in that result.

(f) If the conditions given in (ii), (iv), and (vii) are satisfied,

$$R(\theta)=\frac{(\theta-1)\,(2\lambda L_C-1)}{(\mu+\lambda C(\theta))(\theta+1)-2\theta}. \tag{15}$$

The result in Equation (15) is in agreement with the one given in line 2 after Equation (25), Chaudhry [4], if we first change p, q to λ, μ, respectively, and then put $p=q=1/2$ in that result.

We can discuss the various other combinations, but we leave all these for the prospective reader.

MEAN NUMBER OF PHASES

The mean number of phases is obtained by differentiating the expression for $R(\theta)$ in (11) with respect to θ and then setting $\theta=1$ therein. But this takes an indeterminate form. Therefore to make calculations intelligible we proceed as follows.

Let

$P=$ the numerator of $R(\theta)$ in (11)

$$=(\theta-1)\,[\lambda L_C(2-p_1-p_2)+(p_2-1)]\,[\lambda\alpha C(\theta)-(\mu+\lambda C(\theta))\,(\lambda p_1+\mu(1-p_2)]$$

and

$Q=$ the denominator of $R(\theta)$ in (11)

$$=[\lambda\alpha-\{\lambda p_1+\mu(1-p_2)\}]\,[\{\mu+\lambda C(\theta)\}\{\lambda p_1+\mu(1-p_2)+\lambda C(\theta)(1-p_1-p_2)+p_2\theta \;\; \partial].$$

If we put L_p for the mean number of phases,

$$L_p=\left[\frac{d(P/Q)}{d\theta}\right]_{\theta=1}=\left\{\frac{Q\dfrac{dp}{d\theta}-P\dfrac{dQ}{d\theta}}{Q^2}\right\}_{\theta=1}.$$

This expression is indeterminate for $\theta=1$. Applying L'Hospital's rule twice in order to eliminate the effect of indeterminacy,

$$L_p=\left\{\frac{\dfrac{d^2P}{d\theta^2}\dfrac{dQ}{d\theta}-\dfrac{d^2Q}{d\theta^2}\dfrac{dP}{d\theta}}{2\left(\dfrac{dQ}{d\theta}\right)^2}\right\}_{\theta=1}, \qquad \text{Other terms vanish at } \theta=1,$$

$$=\frac{1}{2}\,\frac{2\lambda L_c[\alpha-(\lambda p_1+\mu(1-p_2))][\lambda L_C(2-p_1-p_2)+(p_2-1)] - [\lambda\alpha-(\lambda p_1+\mu(1-p_2))][\lambda L_{CC}(2-p_1-p_2)+2\lambda L_C\{p_2+\lambda(1-p_1-p_2)L_C\}]}{[\lambda\alpha-(\lambda p_1+\mu(1-p_2))][\lambda(2-p_1-p_2)L_C+(p_2-1)]},$$

where

$$L_{CC}=\sum_{j=2}^{\infty} j(j-1)C_j.$$

IMPORTANT CASES

(a′) If the conditions given in (ii) are satisfied

$$L_p=\lambda\,\frac{-q\{\lambda\alpha-(\lambda p+\mu q)\}L_{CC}+\lambda\{2q+\lambda(p-q)\alpha-(\lambda p+\mu q)\}L_C^2+\{(\lambda p+\mu q)-\alpha(\lambda p+q)\}L_C}{q[\lambda\alpha-(\lambda p+\mu q)][2\lambda L_C-1]}.$$

(b′) If the conditions given in (ii) and (iii) are satisfied,

$$L_p=\frac{\lambda[qL_{CC}-\lambda L_C^2+L_C]}{q(1-2\lambda L_C)}.$$

(c′) If the conditions given in (ii) and (iv) are satisfied,

$$L_p=\lambda\,\frac{-q\{\lambda-2(\lambda p+\mu q)\}L_{CC}+\lambda^2(q-p)L_C^2+(\lambda p+2\mu q-q)L_C}{q(2\lambda L_C-1)[\lambda-2(\lambda p+\mu q)]}.$$

(d′) If the conditions given in (ii), (iii), and (viii) are satisfied $L_C=1$, $L_{CC}=0$, and the mean number of phases becomes the mean queue length with the *departures* correlated, and further that if the system is empty at t_r-0, the unit arriving at t_r cannot depart at t_r. If L_A stands for the mean queue length,

$$L_A=\frac{\lambda\mu}{q(\mu-\lambda)}. \tag{16}$$

The expression on the right side of Equation (16) agrees with that in (15), Chaudhry [3], after we substitute the value of the coefficient of correlation in that result.

(e′) If the conditions given in (ii), (iv), and (viii) are satisfied, the mean number of phases becomes the mean queue length with the *departures* correlated and further that in this case if the system is empty at t_r-0, the unit arriving at t_r can depart at t_r with the probability 1/2. If L_B stands for the mean queue length in this case,

$$L_B=\frac{\lambda\mu(\lambda p+\mu q)}{q[\lambda-2(\lambda p+\mu q)](\lambda-\mu)}. \tag{17}$$

The expression on the right side of Equation (17) is in conformity with that in (35), Chaudhry [3], after we substitute the value of the coefficient of correlation in that expression.

One could proceed further to discuss the various other cases.

VARIANCE

In most of the cases we are interested in the mean number of units in the system and not in the phases and as fluctuations in the number of units waiting in the system can occur, we proceed to find the variance.

In case of a result in Equation (12) if V_A stands for the variance,

$$V_A = R''(1) + R'(1) - [R'(1)]^2 \qquad \text{where } R'(\theta) = \frac{dR(\theta)}{d\theta}$$

$$= L_A + L_A{}^2\left[\frac{2(\lambda q + \mu p) - \mu}{\mu}\right].$$

Proceeding on the above lines we can find variance in case of a result in Equation (13).

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HEURISTIC SOLUTION TO A DISCRETE COLLECTION MODEL

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ABSTRACT

A discrete time Collection Model is formulated, involving the completion of a touring objective on a network with stochastic node states. Heuristic touring strategies are constructed, there being as yet inadequate analytic results for its optimal solution. Effectiveness of the heuristics is assessed by comparing expected tour times under the heuristics with expected tour times given perfect information. A branch and bound algorithm is presented for computing the perfect information tour times.

1. INTRODUCTION

There has recently been formulated a new class of network models (see [3]). These models take a simple network of nodes, edges, and deterministic edge weights, and add the structure of node states with stochastic transitions. An objective to be completed on the network is then specified and strategies are sought to ensure optimal completion of the objective.

One model from the class has been termed the Collection Model. In this model, the nodes may represent physical locations with the edge weights corresponding to travel times between locations. At any time each node occupies one of two possible states, namely, "ready" or "not ready." At some time, each node is not ready. Then, governed by known stochastic laws, each node becomes ready and stays ready. A "tourer" has the objective of visiting each node once when it is ready. Defining success as being at a node when it is ready, the tourer's objective is to achieve a success at each node. Analysis centres upon the selection of decision rules or strategies under which to complete the objective in minimum expected time. The Collection Model may be thought of as a generalization of the travelling salesman problem, where the salesman must not only visit every node, but must visit each node when it is in the proper state. When solved, the Collection Model will find application in such areas as delivery scheduling and machinery allocation (see [3]).

In this paper, heuristic decision rules for a discrete time Collection Model are derived, justified, and evaluated for effectiveness.

2. FORMULATION OF A DISCRETE COLLECTION MODEL

Let $I_+ = \{0, 1, 2, \ldots\}$.

There are n nodes, $n \epsilon I_+$, $n \neq 0$.

(a) The Discrete Time Unit. Time is measured in discrete units, with the tour on the network starting at time zero. Each time the tourer visits a node, *he must stay at that node for one whole time period.* (He may choose to stay for more than one.) This constraint serves to fix the discrete time unit.

(b) The Internode Travel Times. $\mathbf{K} = (K_{ij})$ is the internode travel time matrix, and K_{ij} is the travel time between nodes i and j, $K_{ij} \epsilon I_+$, $i, j = 1, 2, \ldots, n$, $i \neq j$.

(c) The Probability Law. At any time each node is in one of two possible states, ready or not ready. There is some time when each node is not ready. Nodes become ready in accordance with the following stationary probability law:

If node i is not ready at some time t, then at time $t+1$, node i is, (i) not ready with probability p_i, and is (ii) ready with probability $1-p_i$. If node i is ready at some time t, then it is ready at all times s, $s \geq t$.

$\mathbf{p} = (p_1, \ldots, p_n)$ is the vector of probability parameters and p_i is the probability parameter governing the state of node i, $0 \leq p_i \leq 1$, $i = 1, 2, \ldots, n$.

(d) The Pre-Start Times. The tour starts at time zero; however, any given node may begin in the not ready state at some time prior to time zero. At time $-m_i$, node i is known (by the tourer) to be not ready. At time zero, node i has already been subject to change of state, as governed by the probability law, for m_i time periods. $\mathbf{m} = (m_1, \ldots, m_n)$ is the vector of prestart times, and m_i is the prestart time of node i, $m_i \epsilon I_+$, $i = 1, 2, \ldots, n$.

(e) The Lead Times. $\mathbf{d} = (d_1, \ldots, d_n)$ is the vector of lead times, and d_i is the lead time to node i, $d_i \epsilon I_+$, $i = 1, 2, \ldots, n$. d_i is the number of time periods it would take the tourer to initiate the tour at node i (see Figure 1).

(f) Problem Statement. The network for the Collection Model is depicted in Figure 1. It is seen that the parameters of the model are summarized by the 5-tuple $[n, \mathbf{K}, \mathbf{p}, \mathbf{m}, \mathbf{d}]$. A tourer who knows only the values of the entries in the 5-tuple must visit each node once when it is ready. The purpose of analysis is to find a touring strategy under which the objective will be completed in minimum expected time.

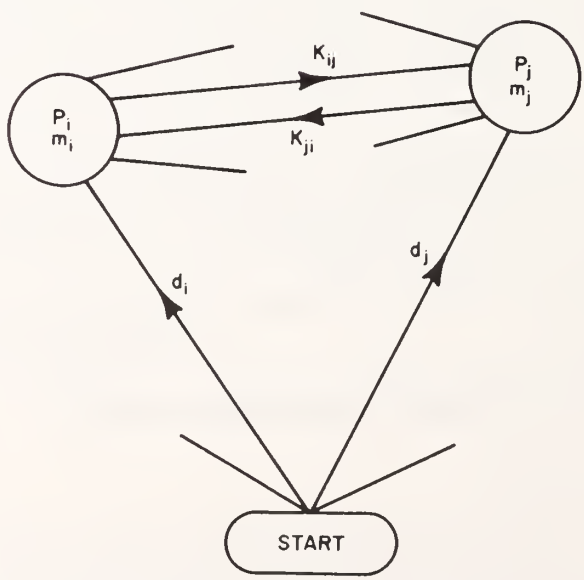

FIGURE 1. The Network for the Collection Model

3. ANALYTIC DETERMINATION OF OPTIMAL STRATEGIES

Certain particular versions of the Discrete Collection Model may be solved for optimal closed-form decision rules. For example, the problem $\mathscr{P}_1 = [n, \mathbf{K}, \mathbf{p}, \mathbf{m}, \mathbf{d}]$ where

$$n = 2$$

$$\mathbf{K} = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}, \qquad K \epsilon I_+,$$

$$\mathbf{p} = (p, p), \qquad p \epsilon [0, 1],$$

and

$$\mathbf{m} = \mathbf{d} = (0, 0)$$

is solved in [2]. This result is generalized in [3] to $\mathscr{P}_2$, where

$$\mathbf{m} = (m_1, m_2), \qquad m_i \epsilon I_+, \qquad i = 1,2,$$

and

$$\mathbf{d} = (d_1, d_2), \qquad d_i \epsilon I_+, \qquad i = 1,2.$$

Also, particular optimal solutions of the problem

$$\mathscr{P}_3 = [n, \mathbf{K}, \mathbf{p}, \mathbf{m}, \mathbf{d}], \text{ where}$$

$$n \epsilon I_+, n \neq 0,$$

$$\mathbf{K} = (0), \qquad \text{the null matrix,}$$

$$\mathbf{p} = (p, \ldots p), \qquad p \epsilon [0, 1],$$

$$\mathbf{m} = (m_1, \ldots, m_n), \qquad m_i \epsilon I_+, \qquad i = 1,2, \ldots, n,$$

$$\mathbf{d} = (0, \ldots, 0)$$

are given in [3].

However, the richly interactive nature of the model, when coupled with dimensionality difficulties, makes the general analytic solution of the Discrete Collection Model elusive. This has led to heuristic considerations. It will be seen that insights gained from the solution of the simple models already mentioned can be applied to the construction of useful heuristics for the more general Discrete Collection Model.

4. PRELIMINARY COMMENTS ABOUT HEURISTIC STRATEGIES

A collection tour may be decomposed into a sequence of decision points. At each point the tourer must decide, (i) which node to visit next, and, (ii) how long to wait there without success. A (heuristic) strategy is specified by providing rules with which to make each decision.

In the remainder of this paper five different heuristics will be derived. The results of computer simulation of these heuristics on two different six-node Collection Models will be presented. The effectiveness of the heuristics will be assessed by comparing tour times under the heuristics to lower bounds on the optimal tour times for these problems, namely, tour times given perfect information. Consideration of the simulation results will lead to a "summary statement"—an overall heuristic to apply to the Discrete Collection Model.

5. DERIVATION OF THE FIRST HEURISTIC, H1

Consider the following constrained shortest path problem: Given a simple network with internode distance matrix $\mathbf{K}=(K_{ij})$, and given a starting node, say i, find the length, B_i, of the shortest path which starts at node i and visits every other node exactly once (not returning to i). Such a path may be called a broken Hamiltonian path, BHP.

The BHP problem may be solved using a variation on Bellman's method for the travelling salesman problem (see [1]).

Defining

$$\mathbf{i}_{m-1}=(i_1, i_2, \ldots, i_{m-1}),$$

$$\mathbf{i}^j_{m-1}=(i_1, i_2, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{m-1}),$$

and

$$S(\mathbf{i}_{m-1})=\{i_1, i_2, \ldots, i_{m-1}\},$$

let $f_m(i;\, \mathbf{i}_{m-1})$ be the length of the BHP from node i through nodes $i_1, i_2, \ldots, i_{m-1}$. The Principle of Optimality yields

$$f_m(i;\, \mathbf{i}_{m-1})= \min_{j\epsilon S(\mathbf{i}_{m-1})} \{K_{i,j}+f_{m-1}(j;\, \mathbf{i}^j_{m-1})\},$$

which may be recursively solved using Dynamic Programming.

These ideas may be applied to the Discrete Collection Model as follows: Augment the collection network with the starting point, called node zero. This produces a new internode time matrix $\mathbf{K}'$ where

$$K'_{ij}=K_{ij}, \qquad i, j=1, 2, \ldots, n,$$

$$K'_{0j}=d_j, \qquad j=1, 2, \ldots, n,$$

$$K'_{i0}=\infty, \qquad i=0, 1, \ldots, n.$$

Find the BHP for the augmented network with node 0 as starting point. This path is called the augmented BHP or ABHP.

The first heuristic, $H1$, is to wait until ready along the ABHP. Thus, under $H1$, the tourer chooses his next node to be a neighbour on the ABHP and waits at each node he visits until he achieves a success at that node.

6. DERIVATION OF THE MSP RULE

Under $H1$ the tourer never leaves a node that is not ready. Let the "termination number" of a node, N, be the number of time periods the tourer will wait at a node without success. (In general, the optimal termination number of any node at any time will depend upon the state of the tour at that time.) $H1$ selects termination numbers of ∞ for each node. The simulation results will indicate that in some cases (for small values of the probability parameter) this is an acceptable strategy. In general, though, the tourer will do better to put an upper bound on the number of time periods he will wait at an unsuccessful node; that is, to choose a finite termination number.

The *MSP* rule is a heuristic rule for selecting termination numbers. The rule is based upon a property of the optimal solution to the problem $\mathscr{P}_3$. When $n=2$ and $n=3$, it is known (and it is conjectured for larger values of n) that the optimal strategy for $\mathscr{P}_3$ is such that in every time period the tourer maximizes his probability of being at a ready node; that is, the probability of success is maximized time period by time period. Extending this idea, under the MSP rule, the tourer *maximizes* his *success probability* over the "short horizon" in the following sense:

Assume that at some time, t, the tourer is about to visit node i where he will wait for N time periods without success (as chosen by the *MSP* rule) and then visit node j, K time units away. Assume that at time t, node j has a prestart time of m_j. This is equivalent to assuming that at time $t-m_j$, node j was known to be not ready. Given that there is no success at node i, the probability of success in the first time period at node j is $1-p_j^{m_j+N+K+1}$. This probability is now averaged over the K travel time periods* and thus N is chosen to be the first positive integer such that

$$\frac{1-p_j^{m_j+N+K+1}}{K+1} > 1-p_i.$$

Let $[Z]_+$ denote the smallest positive integer greater than Z. It follows that

$$N=\left[\frac{\log\,(K-Kp_i-p_i)}{\log\,(p_j)}-m_j-K-1\right]_+ .$$

This is the *MSP* rule. To summarize, under the *MSP* rule the tourer stays at a node that is not ready only until his average probability of success over the short horizon is greater by leaving. The simulation results will indicate that the *MSP* rule is an effective way to terminate the stay at a node that is not ready.

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*If at time t the tourer is in transit between nodes i and j, then the probability of finding a node ready at time t is zero. Thus, to maximize success probability time period by time period would be to *never* to leave an unsuccessful node, except for another node zero travel time periods away. Averaging over the K travel time periods thus maximizes success probability "over the short horizon."

7. DERIVATION OF HEURISTICS 2 THROUGH 5, $H2 \rightarrow H5$

One specifies a heuristic by providing rules to select both the node to visit next and its termination number. Under $H1$ the tourer visits next a neighbour on the *ABHP* and uses infinite termination numbers. Under $H2$ through $H5$ the tourer terminates his stay at a not ready node using the *MSP* rule. The *MSP* rule requires four parameters to generate a termination number: p_i is the probability parameter of the node about to be visited; p_j and m_j are the probability parameter and current prestart time† of the node to be visited after node i, for reasons to be discussed below, K, the internode travel time parameter will be selected in two ways.

$H2$ through $H5$ are constructed by crossing two rules for selecting next node with two rules for selecting the internode travel time parameter to be used in the *MSP* rule.

The first rule for choosing next node is to visit next a neighbour on a Hamiltonian path through the still-active nodes. This is justified by observing that if a node is left without success, then it must be revisited. Given success at all other nodes to be visited, the most efficient path to follow is a Hamiltonian path. The second rule for choosing next node is to visit a neighbour on a *BHP* through the still-active nodes. This is justified in a similar way by observing that if a node is left after having been found ready, then it need not be revisited, and, given success at future nodes, the best path to follow is a *BHP*.

The first rule for choosing an internode travel time value is to make the natural selection $K = K_{ij}$, the travel time to the next node to be visited. The second rule is to choose for K the largest travel time from any still-active node back to the current node. This sometimes pessimistic rule acknowledges the interactive character of the termination decision.

TABLE I

Rules for Selecting Next Node and Internode Travel-time Parameter

Rule for next node		Rule for internode travel-time parameter, K	
1	Neighbour on Hamiltonian Path through still-active nodes	1	Travel time to next node
2	Neighbour on BHP through still-active nodes	2	Largest travel-time from any still-active node back to current node

TABLE II

Heuristics $2 \rightarrow 5$

Heuristic	Rule for next node	Rule for K
$H2$	1	1
$H3$	1	2
$H4$	2	1
$H5$	2	2

†A node with a prestart time greater than zero may well become ready prior to time zero, but cannot be found ready by the tourer until after the start of the tour.

8. ASSESSING THE HEURISTICS USING PERFECT INFORMATION

A common difficulty when dealing with heuristic solutions to problems is how to measure the effectiveness of the heuristics. With the optimal solution unknown, the question of how near to optimal is a heuristic solution may be difficult to answer. For the Discrete Collection Model, however, there is a convenient way to assess the effectiveness of heuristics; that is, to compare expected heuristic tour times with expected tour times given perfect information.

The perfect information collection problem is posed as follows: Let $\mathbf{l}=(l_1, \ldots, l_N)$ be the vector of actual node ready times where l_i is the time period when node i can first be found ready, $l_i=1, 2, \ldots,$ $i=1, 2, \ldots, n$. Given the augmented network $\mathbf{K}'$, and the actual ready times, $\mathbf{l}$, one must find the tour which visits in least time each node when it is ready.

It is shown in [3] that the solution to the perfect information collection problem can be reduced to a search over the discrete space of all node sequences. An algorithm for this search problem will be presented. In preparation, construct the matrix $\mathbf{K}''=(K''_{ij})$ $i, j=0, 1, \ldots, n$. K''_{ij} is the length of the shortest path from node i to node j. (In constructing the matrix $\mathbf{K}''$, one must be careful to charge one time period for every intermediate node visited in going from i to j.)

Given $\mathbf{K}''$, and $\mathbf{I}$, the optimal sequence $i_1^*, i_2^*, \ldots, i_n^*$ is the one which minimizes total tour time when the tourer visits the nodes in the prescribed order, waiting until ready at each node.

The perfect information collection problem has proven to be a good application for the branch and bound search technique, for one can construct an attainable lower bound for the tour time of all tours that start out with the partial node sequence. $i_1, i_2, \ldots, i_L$, $L \leq n$. This bound is given by

$$T(i_L)+B(i_L)+n-L,$$

where $T(i_L)$ is the time that node i_L is found ready and $B(i_L)$ is the length of the *BHP* from i_L through the remaining nodes, computed using $\mathbf{K}''$. Note that if, for any node sequence $a, b, c, \ldots, y, z$, we let $K''(a, z)$ denote the length of the directed path from a to z; that is:

$$K''(a, z)=K''_{a,b}+K''_{b,c}+\ldots,+K''_{y,z},$$

then $T(i_L)$ may be written as:

$$T(i_L)=\max_{1\leq j\leq L}\{l_j+K''(i_j, i_L)+L-j\}.$$

A standard branch and bound algorithm may now be constructed. One first generates a feasible solution and thus obtains an upper bound on the optimal tour time. The space of node sequences is now searched. The branching rule is to pursue a branch by adding still unscheduled nodes until a feasible solution is generated which is better than the current best. If at any time the lower bound on all feasible solutions in that branch is equal to or greater than the value of the current best feasible solution, the branch is terminated and another is continued by adding an unscheduled node to that longest partial node sequence with the smallest lower bound.

Given the Discrete Collection Model $[n, \mathbf{K}, \mathbf{p}, \mathbf{m}, \mathbf{d}]$, the expected tour time given perfect information is estimated by averaging the tour times for a set of perfect information collection problems where

the actual ready times, $\mathbf{l}$, are pseudo-randomly generated according to the probability law:

$$\text{prob}\ (l_i = l) = \begin{cases} 0, & l \leq 0 \\ 1 - p^{m_i+1}, & l = 1 \\ p^{m_i+l-1}(1-p), & l = 2, 3, \ldots \end{cases}, \quad i = 1, 2, \ldots, n.$$

The method just described was used to calculate the expected tour times given perfect information reported in the next section.

9. COMPUTER SIMULATION RESULTS

Tours under $H1$ through $H5$ were simulated for the following six-node Collection Models:

System 1:

$$\mathbf{K} = \begin{bmatrix} 0 & 6 & 6 & 2 & 1 & 1 \\ 6 & 0 & 3 & 1 & 3 & 2 \\ 6 & 3 & 0 & 2 & 6 & 4 \\ 2 & 1 & 2 & 0 & 3 & 3 \\ 2 & 3 & 6 & 3 & 0 & 3 \\ 1 & 2 & 4 & 3 & 3 & 0 \end{bmatrix}$$

$$\mathbf{p} = (p, p, \ldots, p)$$

$$\mathbf{m} = \mathbf{d} = (0, 0, \ldots, 0)$$

System 2:

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{p} = (p, p, \ldots, p)$$

$$\mathbf{m} = \mathbf{d} = (0, 0, \ldots, 0)$$

For each, the probability parameter was varied from 0.55 in steps of 0.05 up to 0.95. Also expected tour times under perfect information were estimated. The simulation results are tabulated below:

System 1:

p	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\frac{1}{1-p}$	2.2	2.5	2.9	3.3	4.0	5.0	6.7	10.0	20.0
$H1$	15.3	15.7	16.3	17.1	18.3	20.3	24.0	31.7	56.2
$H2$	20.3	20.6	21.1	21.8	22.9	24.5	27.5	35.2	52.2
$H3$	20.3	20.6	21.4	21.8	22.9	24.5	27.5	31.6	52.4
$H4$	15.3	15.7	16.3	17.1	18.3	20.4	23.9	31.6	52.6
$H5$	15.3	15.7	16.3	17.1	18.3	20.3	23.8	30.6	52.5
Perf. info.	14.8	15.3	15.3	15.7	16.2	17.2	19.6	26.3	46.6

p	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$\frac{1}{1-p}$	2.2	2.5	2.9	3.3	4.0	5.0	6.7	10.0	20.0
$H1$	9.6	10.1	10.7	11.7	13.1	15.4	19.3	27.4	52.2
$H2$	8.8	8.9	9.4	10.0	11.2	13.3	16.9	25.0	49.9
$H3$	8.3	8.7	9.2	10.1	11.2	13.4	17.0	24.7	49.9
$H4$	8.7	9.1	9.6	10.3	11.5	13.5	17.2	25.1	49.6
$H5$	8.7	9.1	9.5	10.3	11.4	13.5	17.2	24.9	50.4
Perf. info.	7.0	8.0	8.8	9.9	10.9	12.9	16.7	23.4	45.8

The results for System 1 indicate that for small values of p the choice of which node to visit next is important. When p is large (see $p = 0.95$), the question of how long to wait without success is more important. Instead of the absolute value of p, it is clear that a more meaningful indicator of which decision is more critical will be the relative values of the expected number of time periods to success at any node and internode distance. The results for System 2 indicate that when the internode distances are relatively small, the question of how long to wait at any node without success dominates.

Comparison of the heuristic results with the perfect information figures, which are unattainable lower bounds on optimal tour times, indicates that the heuristics are indeed effective.

10. CONCLUSION – A SUMMARY HEURISTIC

The simulation results lend themselves to the following semi-quantitative summary heuristic. Although this heuristic has not yet been thoroughly tested, we believe it will produce efficient solutions to Discrete Collection Problems.

(i) Select the first node to be a node adjacent to the starting point on the *ABHP*.

(ii) Select subsequent nodes as follows:

(a) If the current node is left without success, choose a neighbor on a Hamiltonian path through the still-active nodes.

(b) If the current node is left with success, choose a neighbor on a *BHP* through the still-active nodes.

(iii) Terminate the stay at an unsuccessful node in accordance with the *MSP* rule using an appropriate K value.

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EXPECTATION OF CONTRACT INCENTIVES*

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The application of statistical expectation to risk density functions and fee/incentive-element relationships is shown to be useful in structuring contract incentives. A mathematical procedure for calculating the expected value of fee for a given risk/incentive arrangement is described along with cost examples and related sensitivity analyses. The structuring of equivalent incentives is demonstrated by the use of the contracting procedure used for procuring the C–5A aircraft.

1. INTRODUCTION

The purpose of contract incentives is to motivate the contractor to produce a system that will meet or surpass performance goals, on or before a target date, and within or at target cost. Much of the work aimed at developing and improving methods of evaluating and structuring incentives has concentrated on NASA and DOD system development and procurement. Scarborough [7] discusses the advantages of expressing incentives in terms of system cost/effectiveness measures and illustrates the use of analytical techniques in evaluating incentives in terms of customer and contractor strategies. A computer-aided technique has been developed for evaluating and structuring incentives such that the fee received by the contractor is dependent on the customer's combined value of cost, performance, and delivery outcome [2, 4, 9]. Effort by Belden [1] and Fisher [3] has been directed toward determining the effectiveness of contract incentives through empirical analysis. The purpose of this paper is to show the potential usefulness of the *expected value of fee* during the formulation of incentives.

2. DEFINITIONS AND METHODOLOGY

This paper will only illustrate the use of the *expected value of fee* with respect to cost incentives although the basic methodology is also applicable to performance and schedule incentives.

Two basic incentive forms are cost plus incentive fee (*CPIF*) and fixed-price incentive firm (*FPIF*). The FPIF incentive form is characterized by the following items.

(1) Target Cost (*TC*)—that cost for which there is considered at the time of contract negotiation a 50 percent chance of a cost overrun or underrun.

(2) Target Fee (*TF*)—a percentage of target cost which represents a fair profit for the work at target cost.

(3) Cost Ceiling (*CC*)—that dollar amount which represents the customer's maximum liability.

(4) Sharing Line or Formula (*SL*)—the fee/cost arrangement which dictates the contractor's fee for a given cost outcome (e.g., an 85/15 share line means that the contractor will receive target fee plus 15 percent of any underrun or minus 15 percent of any overrun). A description of a *CPIF* incentive

*This paper is based on work documented in "Cost Research Report—1969: Aircraft Risk Assessment and Its Relationship to Contract Incentive Planning," Convair Aerospace Division of General Dynamics, ERR–FW–989 (31 Dec. 1969).

also requires a target cost, target fee, and sharing formula, but is distinguished from the *FPIF* form by not having a cost ceiling and having instead the following items.

(1) Maximum Fee ($F_{\max}$)—the maximum fee which can be received by the contractor for outstanding success in cost control.

(2) Minimum Fee ($F_{\min}$)—the minimum fee which can be received by the contractor regardless of the degree to which the final program cost exceeds the target cost. Obviously, a *CPIF* becomes a cost-plus-fixed-fee arrangement for all cost outcomes that exceed the cost associated with the $F_{\min}$.

The true cost may be represented as a random variable C with distribution function $F_C(c)$. Letting $h(C)$ represent fee as a function of cost, expected fee is given as the Stieltjes integral

$$E[h(C)]=\int_c h(c)dF_C(c)$$

for any integrable fee function $h(c)$. For the purposes of this paper, $h(c)$ is expressed in either of two commonly used forms, i.e., a constant multiplied by the complement of a normal cumulative distribution function, $k[1-N(c;\ \mu,\ \sigma)]$, or a linear piecewise continuous function; and $f(c)$ is either a normal probability density function, i.e., $n(c;\ \mu,\ \sigma)$, or a triangular probability density function, $t(c;\ a,\ m,\ b)$, where a and b are the end points and m is the mode.

The expected fee, $E[h(c)]$, for the share relationship, $h(c)$, and cost probability density function, $f(c)$, depicted in Figure 1, is equal to target fee, i.e., $TF=\$10$ million. In fact, if $h(c)$ can be expressed as an odd function about TC and the mean, μ, of a symmetrical cost density function, $f(c)$, is equal to TC, then

$$E[h(c)]=\int_{-\infty}^{\infty} h(c)f(c)dc=TF.$$

In this paper, this is referred to as the odd function property.

3. PARAMETRIC ANALYSIS USING TRIANGULAR FORM OF $f(c)$

To expedite computations, the triangular form of $f(c)$ was used to perform the parametric analysis.

Standard Deviation, Sharing Formula, and Mean

To determine the sensitivity of $E[h(c)]$ to changes in the standard deviation, let us assume that a contractor has performed a cost/risk analysis and obtained the symmetrical risk density function, $f(c)=t(c;\ 80,\ 100,\ 120)$, shown in Figure 2. Also assume that a 50/50 customer/contractor share line is proposed in either a *CPIF* or *FPIF* form. Assume that no cost ceiling is proposed for the *FPIF* form. If the contractor is confident in the mean of $f(c)$, to what extent would an error in the standard deviation affect the expected value of fee?

Application of the odd function property shows that the expected value of fee, $E[h(c)]$, is independent of a change in any one of the following if the other two remain constant: standard deviation, form of incentive, or slope of the share line. However, singular changes in the target fee or target cost will affect $E[h(c)]$.

The remainder of the sensitivity analysis only considers the *FPIF* form.

What if the contractor feels confident that he has accurately estimated the range of the risk density function, but is uncertain about the correct location of its mean or target cost? By using the

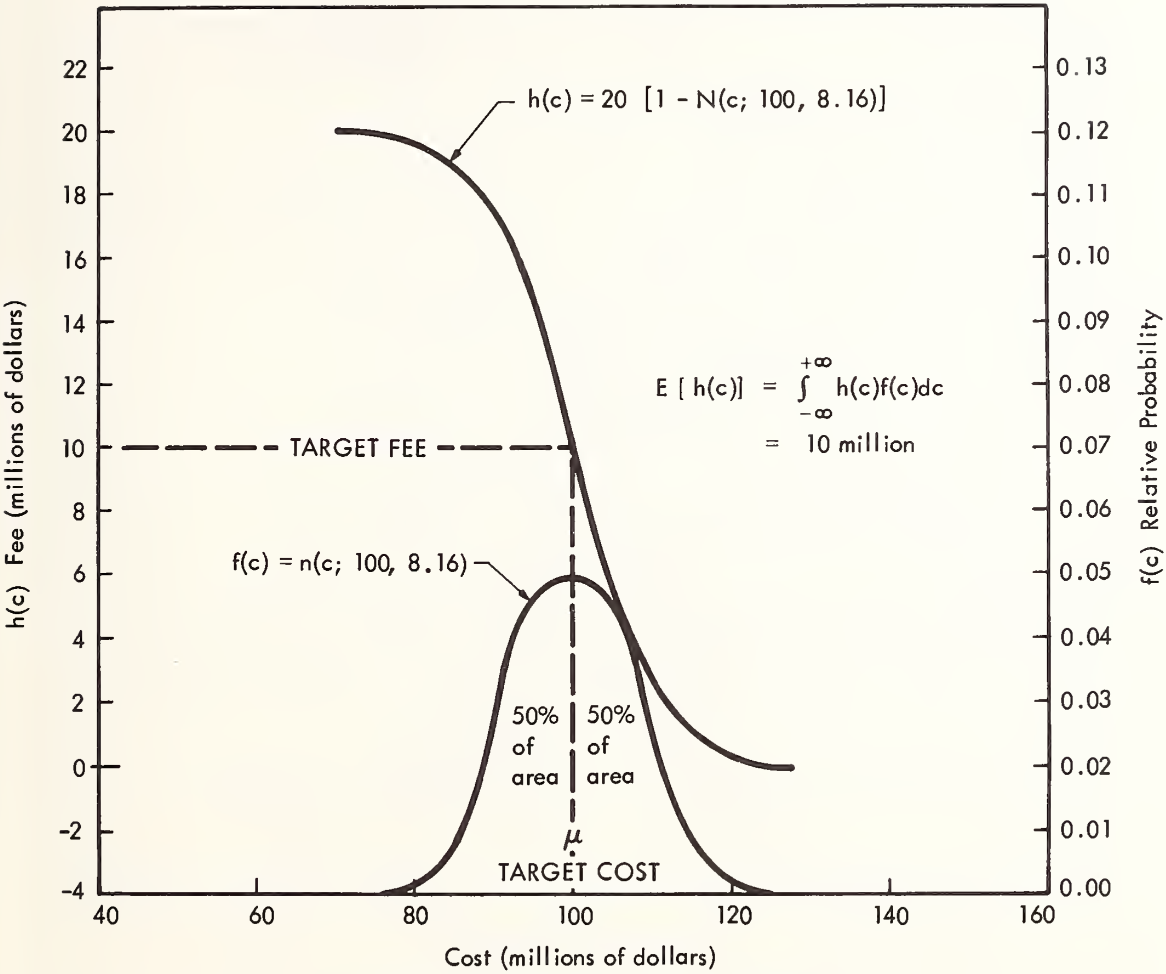

FIGURE 1. Example of incentive expectation

risk density function where $TC=100$ as the baseline and sharing formula $h(c)_{50}$ for all c (no cost ceiling) depicted in Figure 3, it is obvious that for every 10 percent negative or positive change in target cost there is a corresponding 50 percent positive or negative change, respectively, in the expected value of fee. This result is shown in Figure 4.

Imposing a cost ceiling on the *FPIF* arrangement equal to 120 percent of target cost requires $h(c)=h(c)_{100}=120-c$ for $c>120$. As depicted in Figure 4, the result of the sensitivity analysis using this cost ceiling shows a larger, disproportionate percent change in $E[h(c)]$ for each 10 percent positive increase in target cost. Thus, in this example, an underestimated target cost can significantly affect $E[h(c)]$.

Positive Skewness

The effect of positive skewness on $E[h(c)]$ would be of interest to the contractor in structuring an *FPIF* incentive if he felt confident that he had correctly estimated the best, a, and most likely, m, cost outcomes, but was unsure as to the cost that may be the worst possible outcome, b. This condition is represented in Figure 5 where $f(c)=t(c;\ 80,\ 100,\ 120)$ is the baseline and the worst possible

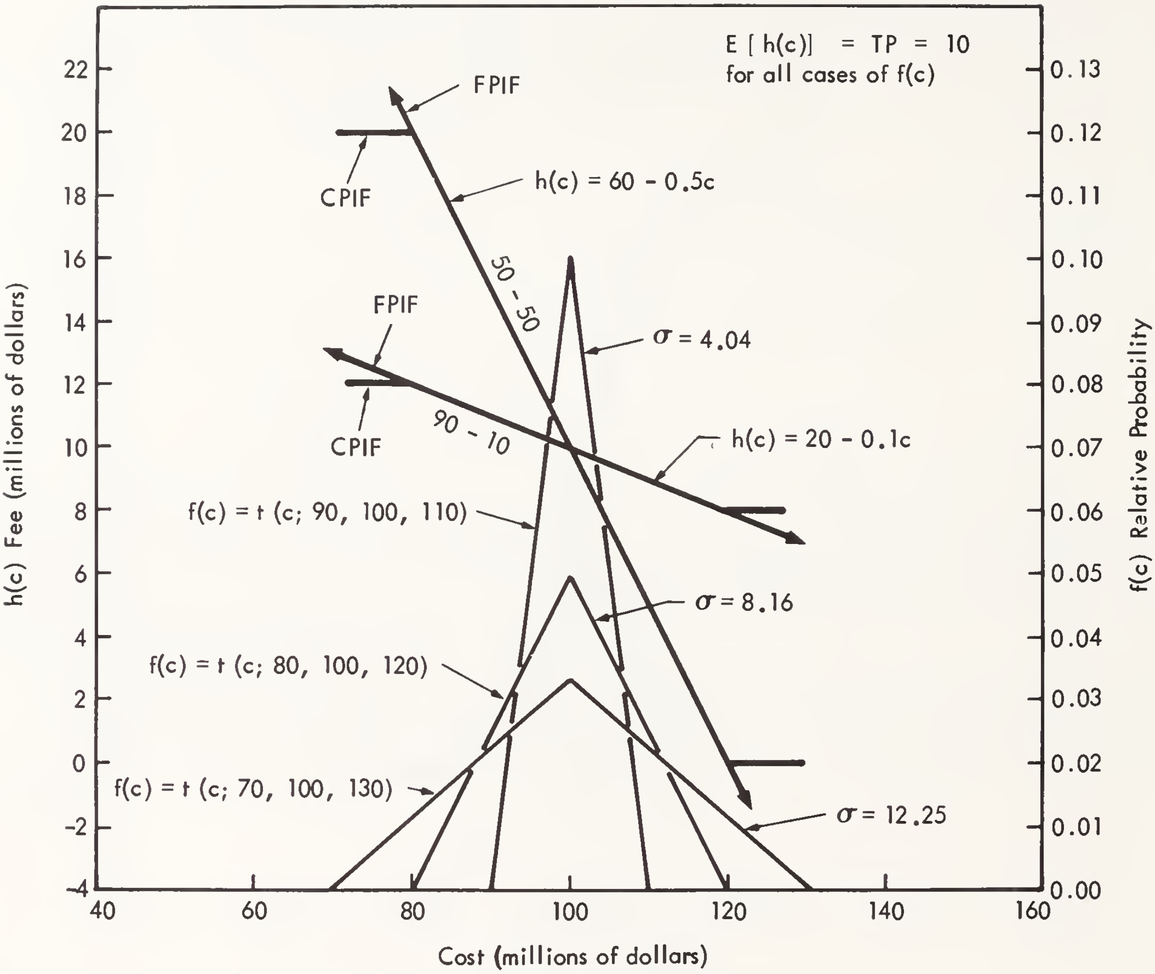

FIGURE 2. Variation in standard deviation and share line

cost outcome, b, is consecutively incremented by \$10 million. As depicted in Figure 6, the analysis of this condition shows that percentage increases in the worst possible outcome, b, produce larger, disproportionate negative percentage changes in $E[h(c)]$ when a cost ceiling is incorporated. As one would expect, the comparison of the results shown in Figure 4 with that shown in Figure 6 indicates that $E[h(c)]$ is considerably more sensitive to errors in the target cost than errors in b.

Incentives with Equivalent Expectations

Another interesting and practical use of the expected value property is in the formulation of an incentive which has the same expected value as another given incentive. For illustration, incentive equivalence will be applied to the incentive contracting procedure used for acquiring the C–5A transport.

Each company competing for the C–5A contract (covering RDT&E, 57 total systems, and related support) was required to propose cost incentives on the basis of three alternative *FPIF* formulas—85/15 over target and 50/50 under, 70/30 over target and 50/50 under, and a flexible incentive with an initial share ratio of 85/15 over and under target. Under each arrangement, target fee was required to equal

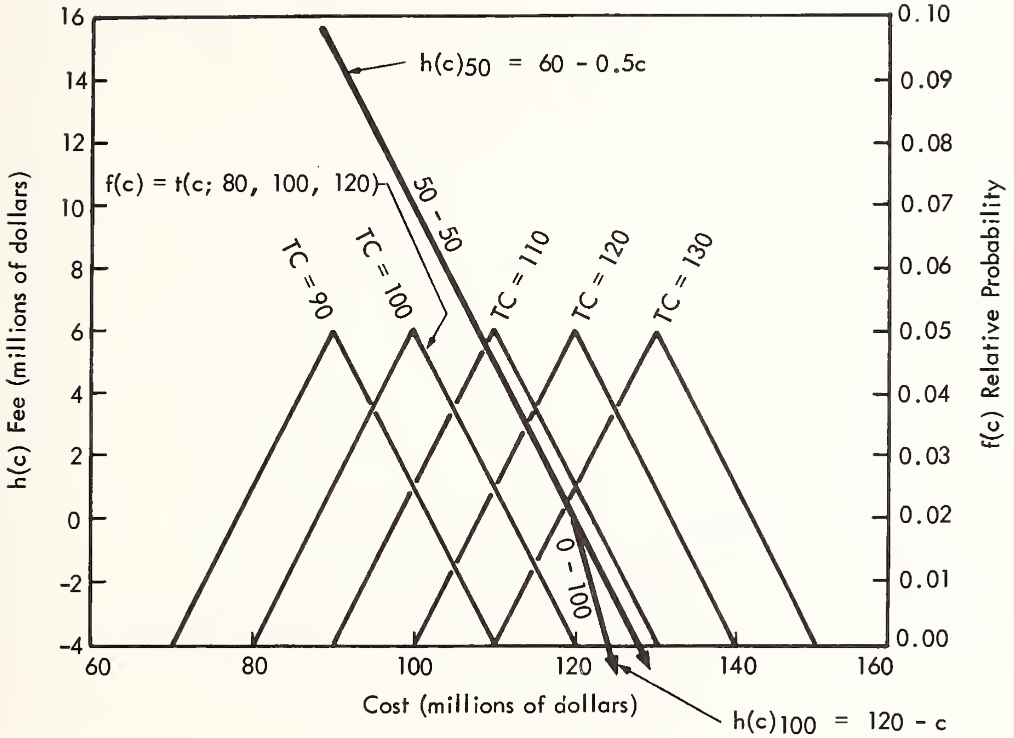

FIGURE 3. Variation in mean

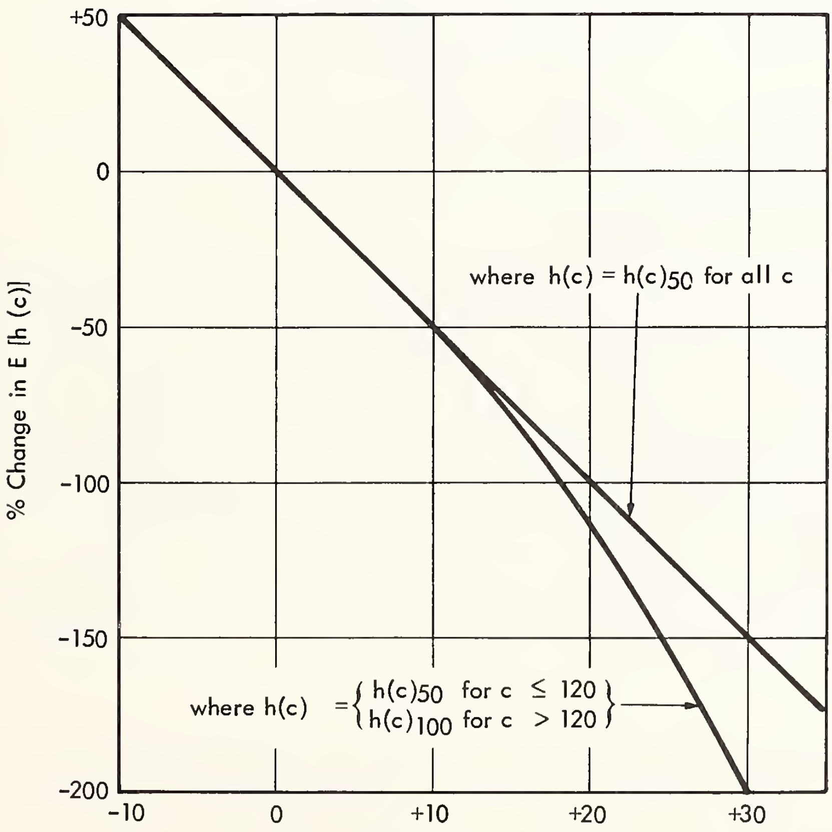

FIGURE 4. Effect of mean on E[$h(c)$]

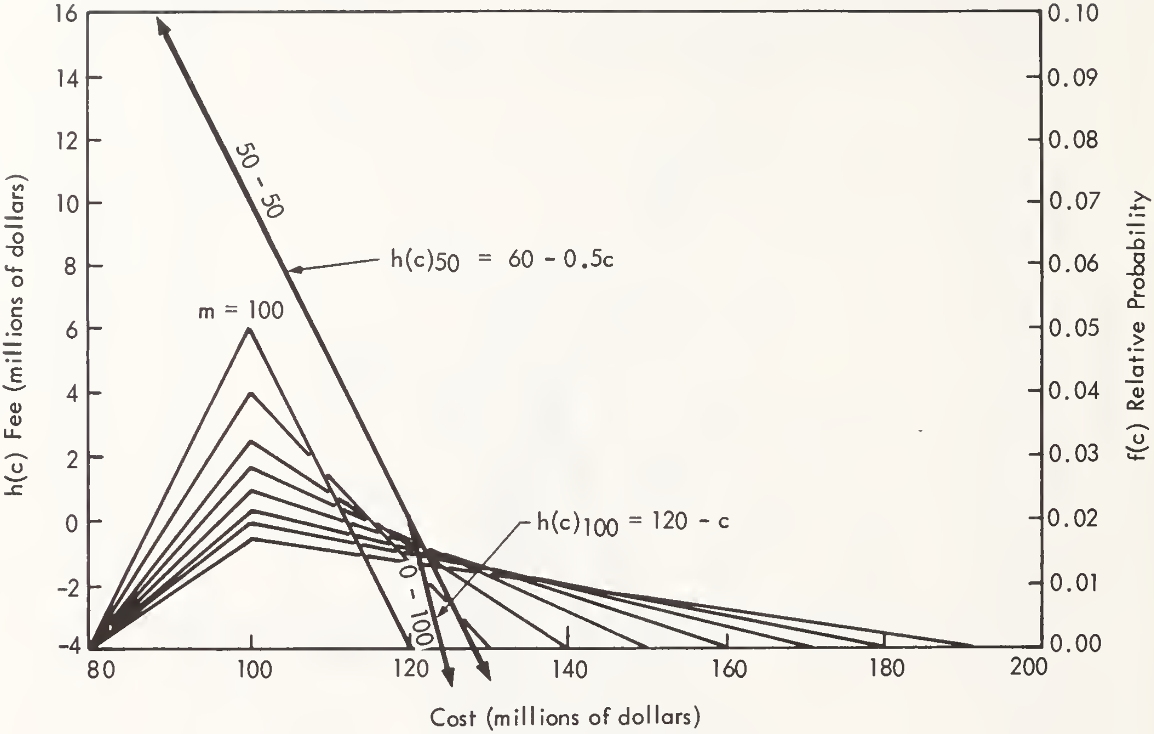

FIGURE 5. Variation in b

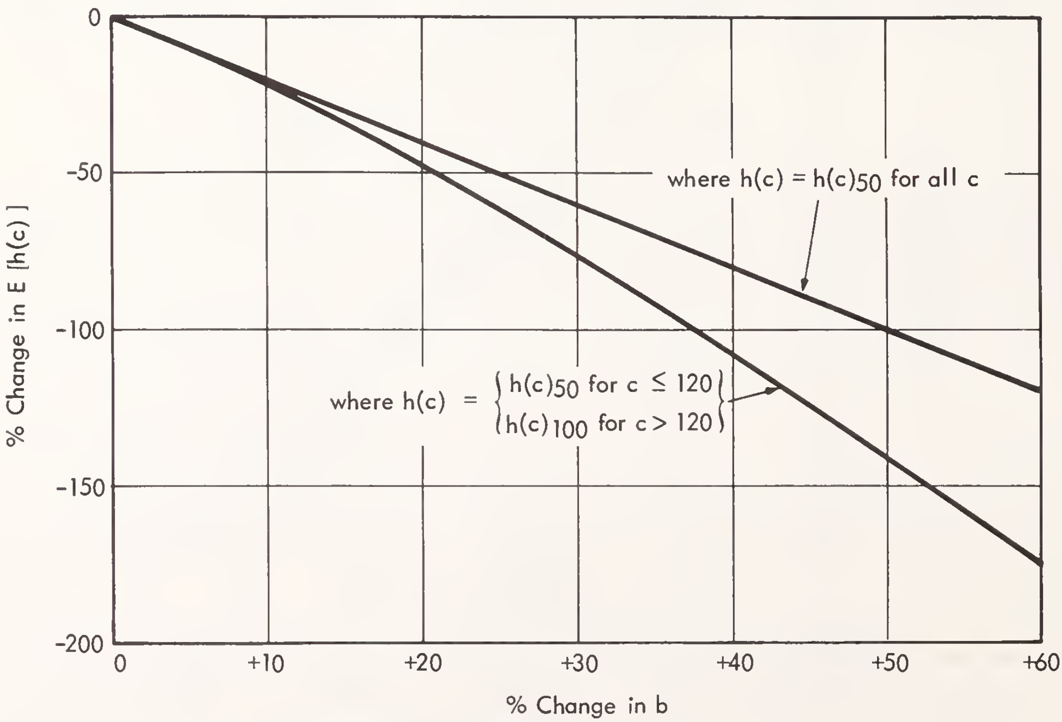

FIGURE 6. Effect of b on $E[h(c)]$

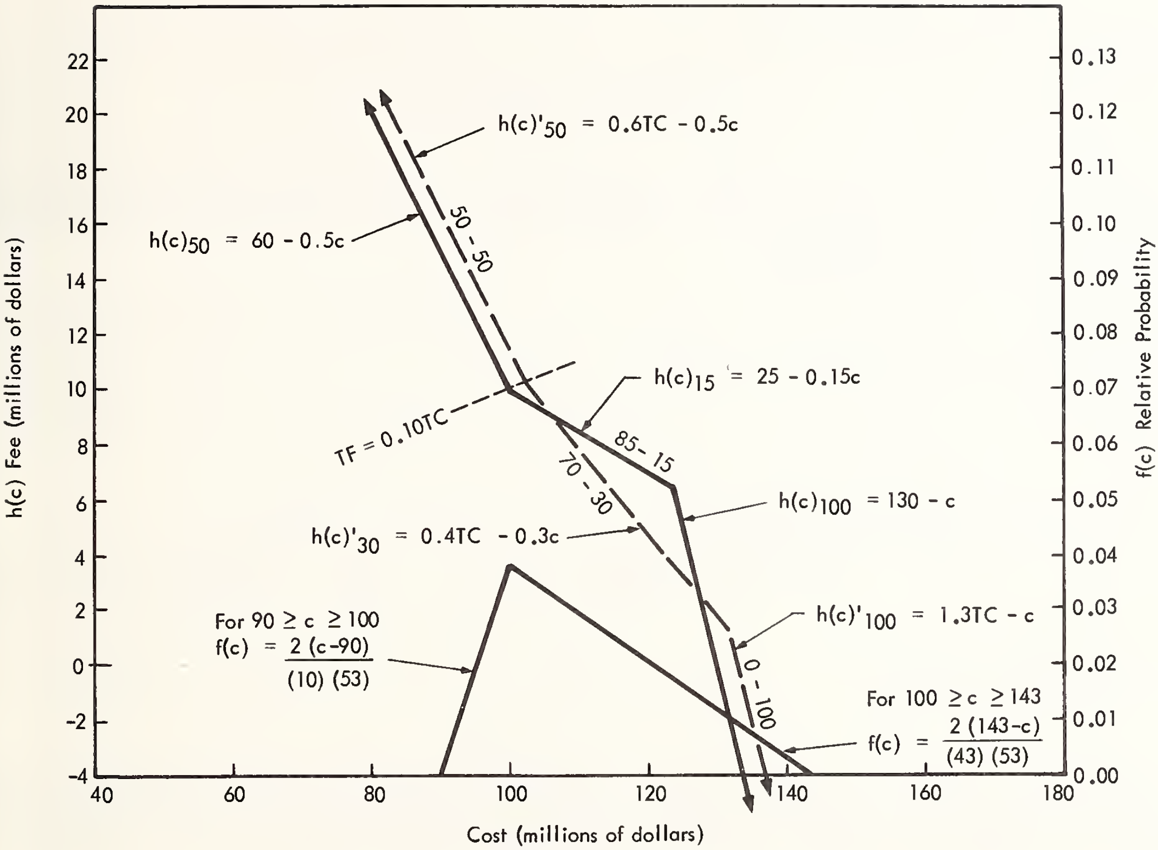

FIGURE 7. Incentives with equivalent expectations

10 percent of target cost and the cost ceiling to equal 130 percent of target cost. Under these ground rules, each bidder was required to set a target cost for each alternative incentive arrangement. After the contracts were signed and submitted, the Air Force would then select the winning company and one of his proposed cost arrangements for accomplishing the work, [8].

Let us assume that one of the competing companies performed an analysis which established that the cost risk was distributed by the positively skewed, triangular density function, $f(x)$, shown in Figure 7. Thus, the best possible cost outcome is \$90 million, the most likely outcome is \$100 million, and the worst possible is \$143 million. Let us further assume that after careful consideration of competition, probability versus profit levels, expected value of fee, and other factors the company set the target cost for the 85/15 over and 50/50 under arrangement at \$100 million (solid line cost incentive shown in Figure 7). The mathematical expression of the expected value of fee, $E[h(c)]$, for this incentive is

$$E[h(c)] = \int_{90}^{100} (60 - 0.5c) \frac{2(c-90)}{(10)(53)} dc + \int_{100}^{123.53} (25 - 0.15c) \frac{2(143-c)}{(43)(53)} dc$$

$$+ \int_{123.53}^{143} (130 - c) \frac{2(143-c)}{(43)(53)} dc.$$

Piecewise integration and evaluation gives

$$E[h(c)] = 7.6524 \text{ million.}$$

For the purpose at hand, the flexible incentive arrangement will be disregarded along with such potential factors as the company's desire to influence the Air Force's selection by intentionally biasing one of the incentive arrangements.

Thus, the company's problem is reduced to that of selecting a target cost for the 70/30 over and 50/50 under arrangement so that the resulting incentive will have the same expected value of fee as the 85/15 over and 50/50 under case. This can be accomplished by first mathematically expressing the 70/30–50/50 arrangement in terms of cost and target cost and setting the result equal to 7.6524; that is,

$$E[h(c)]' = \int_{90}^{100} (0.6TC - 0.5c)\,\frac{2(c-90)}{(10)(53)}\,dc + \int_{100}^{TC} (0.6TC - 0.5c)\,\frac{2(143-c)}{(43)(53)}\,dc$$
$$+ \int_{TC}^{7/9TC} (0.4TC - 0.3c)\,\frac{2(143-c)}{(43)(53)}\,dc + \int_{7/9TC}^{143} (1.3TC - c)\,\frac{2(143-c)}{(43)(53)}\,dc = 7.6524.$$

Thus one obtains

$$TC = 102.46 \text{ million.}$$

By setting the target cost at \$102.46 million for the 70/30–50/50 arrangement, the resulting incentive (dashed-line incentive in Figure 7) will have the same expected value of fee as the 85/15–50/50 incentive. Knowing the target cost, the line segments of the 70/30–50/50 incentive are given by

$$h_{50}(c)' = 61.48 - 0.5c \qquad \text{for } 90 \leq c \leq 102.46,$$

$$h_{30}(c)' = 40.98 - 0.3c \qquad \text{for } 102.46 \leq c \leq 131.73, \text{ and}$$

$$h_{100}(c)' = 133.2 - c \qquad \text{for } 131.75 \leq c \leq 143.$$

4. CONCLUSIONS

Although statistical expectation was shown to be potentially useful in structuring incentive contracts, it should be emphasized that the expected value of fee is only one of several parameters to consider in the evaluation of alternative incentive arrangements. For instance, although the expected value of fee was shown to be independent of the slope of the sharing formula in the example depicted in Figure 2, a risk-avoiding company may prefer the 90/10 *FPIF* share line to the 50/50 *FPIF* share line depicted in Figure 7 since the former gives a higher minimum fee. In the C–5A procurement example the company's desire to influence the Air Force's selection between the two cost incentive arrangements could have overshadowed the desirability of expectation equivalence, e.g., by intentionally setting the target cost of the 70/30–50/50 arrangement higher than required for $E[h(c)]$ equivalency, the company could intentionally bias the 70/30–50/50 arrangement in favor of the 85/15–50/50.

The ultimate usefulness of statistical expectation essentially rests on the ability to accurately quantify risk. Even a reasonable assessment of the cost, schedule, or technical uncertainty associated with a large, complex program is difficult at best. However, explicitly defined expressions, such as those by Marshall [5], should aid in risk quantification. It should be recognized that when agreement has been reached by the customer and contractor on the structure of a contract incentive, risk has been at least subjectively defined by both parties. Thus, at the minimum, the transformation of this subjective expertise into risk distributions will provide the means for gaining insight into the sensitivity of incentives to parameter errors.

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A NOTE COMPARING GLOVER'S AND YOUNG'S SIMPLIFIED PRIMAL ALGORITHMS

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ABSTRACT

Glover and Young, References [4] and [8], respectively, present convergent primal integer programming algorithms. The algorithm outlined in Young's paper (which was deliberately specialized) is shown to be a special case of the Glover algorithm under his acceptable source row selection, Rule 1.

INTRODUCTION

As Gomory's dual all-integer integer programming algorithm (see Ref. [5]) is an extension of the dual simplex method, primal all-integer integer programming algorithms are an extension of the primal simplex method. Specifically, the procedure requires an all integer primal feasible initial tableau. It adds a Gomory (all integer) cut at each iteration, starting with the first, so as to maintain an all integer tableau and primal feasibility. When dual feasibility is reached the tableau is integer optimal.

The "Direct Algorithm", a primal technique, was first described by Ben-Isreal and Charnes [2]. That algorithm, however, frequently required the solution of a sometimes difficult (integer programming) "auxiliary problem." Richard Young [7] proposed a different primal algorithm, labelled the "Rudimentary Primal Algorithm" (RPA), which avoided the auxiliary problem. However, the technique was somewhat complicated and difficult to implement. Fred Glover [3] proposed a pseudo primal-dual algorithm which utilizes Gomory's dual all integer method (Ref. [5]) with a variation of Young's primal all integer technique (Ref. [7]). Subsequently, Glover [4] and Young [8] drawing from their (and each others) work, developed simplified (finite) primal algorithms (the "Simplified Primal Algorithm"—SPA). The algorithms outlined in these papers are essentially built on the same foundations and, in fact, often overlap. There are some differences. For one thing they employ different strategies and tableau format. More importantly, however, is in defining the "reference equation" (introduced to find the pivot column) and source row (from which the Gomory cut or pivot row is generated). Although both papers present the general properties necessary of the reference row and the source row selection rule, Young, for ease of exposition and implementation, outlines an algorithm about a particular reference row and source row selection criterion. Glover, on the other hand, does not specialize these—thus creating a more general approach. We discuss these methods.

Essentially the Simplified Primal Algorithm (SPA) contains the following steps:

(a) adjoin a reference equation to the bottom of the simplex tableau whose coefficients satisfy certain properties.

(b) select the pivot column by utilizing the magnitude of the coefficients in the reference equation.

(c) select a source row from those that will maintain primal feasibility according to an acceptable row selection rule. Derive a Gomory all integer cut which serves as the pivot row.

(d) perform a primal simplex pivot step and either go to (a) or (b) (which depends on the algorithm, i.e., Young's or Glover's).

ALGORITHM STRATEGIES, TABLEAU FORMAT

Suppose the source row is

$$x_i = a_{io} + \sum_j a_{ij}(-x_j), \tag{1}$$

then the Gomory cut or pivot row is

$$s = \left[\frac{a_{io}}{a_{ip}}\right] + \sum_j \left[\frac{a_{ij}}{a_{ip}}\right](-x_j) \geqslant 0, \tag{2}$$

where $[y]$ means the largest integer $\leqslant y$, p is the pivot column index, $a_{ip} > 0$, and the x_j's are the nonbasic variables. Young chooses to distinguish between iterations at which $\left[\frac{a_{io}}{a_{ip}}\right] = 0$ (equivalently $a_{io} < a_{ip}$), labelled stationary cycles, and those at which $\left[\frac{a_{io}}{a_{ip}}\right] \geqslant 1$, referred to as transition cycles. During transition cycles the Rudimentary Primal Algorithm is followed. (The RPA, which does not necessarily converge, is the primal simplex algorithm except for the pivot row which is a Gomory cut derived from any row that will maintain primal feasibility. Note that the usual "pivot row" selected by the simplex algorithm can serve as the source row.) When $\left[\frac{a_{io}}{a_{ip}}\right] = 0$, the Simplified Primal Algorithm is employed. At the beginning of each sequence of stationary cycles a new reference equation is adjoined and updated. Upon termination of the sequence it is dropped. Glover, on the other hand, always employs the SPA and thus uses only one reference row. (The results of Ref. [1] suggest that the additional time required to distinguish between iteration types and to revise the reference row is usually not worthwhile when compared with the reduction in the total number of iterations. There may, however, exist problem classes where the distinction is better. In particular, the class of problems outlined in Ref. [6], which does not converge under the RPA, will converge if this strategy is employed. Also, successive reference rows might accelerate convergence; e.g., by introducing tighter bounds on variables.)

As a minor point, Glover chooses to maintain the identity equations $x_j = -(-x_j)$ for each original nonbasic variable. Young introduces an identity row only when an original nonbasic variable is about to become basic. In either case, a unique pivot column determination is assured.

THE REFERENCE ROW

Write the reference row as

$$x_L = a_{L0} + \sum_j a_{Lj}(-x_j) \geqslant 0, \tag{3}$$

where x_L is a nonnegative slack variable. Then, besides having rational coefficients and not eliminating

any integer feasible solutions, the coefficients in (3) must also satisfy

(I) $$\alpha_j \overset{\mathcal{L}}{<} 0 \Rightarrow a_{Lj} > 0 (j \neq 0)$$

and

(II) $$a_{Lj} < 0 \Rightarrow \frac{\alpha_j}{a_{Lj}} \overset{\mathcal{L}}{<} \frac{\alpha_p}{a_{Lp}}$$

for every $a_{Lj} (j \neq 0)$ different from 0. (α_j denotes the jth column of the tableau.) Properties (I) and (II) are introduced by Glover [4] and Young [8]. In the former paper the reference rows are discussed within the framework of these properties. Young, however, chooses to introduce at the beginning of each sequence of stationary cycles an Equation (3) which has $a_{Lj}=1$ for every $j \neq 0$ and a_{L0} equal to an upper bound on the $\sum_j x_j$. (Note that (I) and (II) are satisfied.)

ACCEPTABLE SOURCE ROW SELECTION RULES

Suppose the pivot column index is p. (The pivot column is selected so that $\frac{\alpha_p}{a_{Lp}} \overset{\mathcal{L}}{<} \frac{\alpha_j}{a_{Lj}}$ for every $j \neq 0$ with $a_{Lj} > 0$.) Then any row (1) which is in $V(p) = \left\{ i \middle| 0 \leqslant \left[\frac{a_{i0}}{a_{ip}} \right] \leqslant \theta_p, a_{ip} > 0, i \neq 0 \right\}$, where $\theta_p = \underset{a_{ip} > 0, i \neq 0}{\text{minimum}} \left\{ \frac{a_{i0}}{a_{ip}} \right\}$, can be used to generate on all integer cut (2) or pivot row which will maintain primal feasibility. To support a convergent algorithm, however, it is necessary to choose a source row from $V(p)$ according to an "acceptable rule." In particular, Young indicates that any rule containing implication (4) is acceptable.

(4) For any row i tableaux must occur at finite intervals in which $a_{ip} \leqslant a_{i0}$.

(A more general criterion for source row selection is given but not used). Glover, on the other hand, gives three specific (and different) rules. The first one is

RULE 1: Choose any row from $V(p)$ that ensures $a_{Lp} \leqslant a_{L0}$ at finite intervals, and which also periodically reduces $\frac{a_{ip}}{a_{Lp}}$ for the smallest $i \geqslant 1$ such that $\frac{a_{ip}}{a_{Lp}} > a_{i0}$. We shall show, as suggested by Glover [4], that rules which contain implication (4) will automatically satisfy Rule 1. To do this note that when $\frac{a_{ip}}{a_{Lp}} > a_{i0}$ we must have $a_{ip} > 0$ (since $a_{i0} \geqslant 0$ and $a_{Lp} > 0$); and for $a_{ip} > 0$, $\frac{a_{ip}}{a_{Lp}} > a_{i0}$ implies $a_{ip} > a_{i0}$. $\left(\text{This means that } \left[\frac{a_{i0}}{a_{ip}}\right] = 0, \text{ or } i \text{ is in } V(p).\right)$ Thus, a rule satisfying (4) must eventually select the smallest $i \geqslant 1$ which has $\frac{a_{ip}}{a_{Lp}} > a_{i0}$. This rule will then ensure for this i that a_{ip} becomes $\leqslant a_{i0}$. But when $a_{ip} \leqslant a_{i0}$, we have $\frac{a_{ip}}{a_{Lp}} \leqslant a_{i0}$ (since $a_{Lp} \geqslant 1$); or $\frac{a_{ip}}{a_{Lp}}$ has been reduced. Hence, the requirements of Rule 1 are met. Note that for $i = L$ the rule and implication coincide.

COMMENTS

1. We have just demonstrated that Rule 1 contains weaker requirements than those having implication (4). Therefore, it suffices to show that Rule 1 supports a finite algorithm. Then, other rules inferring (4) will automatically guarantee convergence.

2. Except for distinguishing between stationary and transition cycles the sample algorithm outlined by Young can be thought of as a special case of Glover's. It is a simple matter to modify the convergence proof under Rule 1 (see Ref. [4]) to allow for this strategy.

3. It should be made very clear that, as previously mentioned, Young's algorithm is purposely specialized. In fact, the reference row is described as "a convenient special case" and the most general requirements are explicitly given. Also, the source row criterion is (implicitly) characterized as special and an explicit description is given of a more general criterion for source row selection. Thus, the algorithm outlined in Reference [8] is not only a special case of the one in Reference [4], but it is also a special case of a more general algorithm which was deliberately not outlined, but nevertheless proposed, by Young.

ACKNOWLEDGEMENT

I must express my sincere thanks to Professor Richard Young for commenting on an earlier version of this paper which, most unfortunately, did not take into account the last section of his paper ("Extensions, Alternatives, Generalizations"); thus, comment 3 (which is essentially a quote) did not emerge. Also for correcting some points on the development of the primal integer programming and for the remarks concerning tighter upper bounds when introducing new reference rows.

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A NOTE ON THE EXTENSION OF A RESULT ON SCHEDULING WITH SECONDARY CRITERIA*

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ABSTRACT

A result of Smith previously published in this journal [3], on the use of secondary criterion in scheduling problems is extended, and an example presented.

Smith [3] (also see [1] and [2]) has presented a result for the n job, one machine scheduling problem which allows the minimization of means (Weighted) flow time if maximum tardiness remains zero. It is possible to relax the restriction that maximum tardiness remains zero.

Following the notation of Conway, et al. [1], let p_i and d_i be the processing time and due date of job i. The maximum tardiness we define as $T_{\max}$,

$$T_{\max} = \max\left[0.\ \max_i\left(\sum_{j\leq i} p_{[j]} - d_{[j]}\right)\right],$$

where $[i]$ denotes the job processed in the ith position. Therefore, we can write an extended version of Smith's result as:

THEOREM: When one schedules an n job, one machine problem, there is an ordering of the jobs with job K in the last position which minimizes the mean flow time, subject to the condition that the maximum tardiness is not increased if, and only if

(i) $$\sum_{i=1}^{n} p_i - d_K \leq T_{\max},$$

and

(ii) $$p_K \geq p_i \qquad \text{for all } i \text{ such that } \sum_{j=1}^{n} p_j - d_i \leq T_{\max}.$$

(Note that $T_{\max} = 0$ is equivalent to Smith's result).

PROOF: The proof is similar to that of Smith [3]. Let the jobs $1, 2, \ldots, n$ be ordered according to increasing due dates which will minimize the maximum tardiness (see [1] and [3]). Thus job n satisfies condition (i). Let K be another job satisfying (i) and such that $p_K > p_n$. Interchanging jobs K and n

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and examining the definition of mean flow times [3], $\bar{F} = (1/n) \sum_{i=1}^{n} (n - i + 1)p_i$, indicates that $\bar{F}$ will decrease by

$$p_n + (n - K + 1)p_K - [p_K + (n - K + 1)p_n] = (n - K)\ (p_K - p_n) > 0.$$

Alternatively, if we try to shift another jog J into the last position that doesn't satisfy (i), then T_{max} will be increased. If J doesn't satisfy (i), then the sum of flow time will be increased by $(n - J)$ $(p_n - p_J)$.

As an example consider the following problem:

job i	1	2	3	4
p_i	4	3	2	2
d_i	3	4	5	6

The solution follows similarly to that given by Smith [3]. Ordering the jobs by increasing due dates, we find

$$\bar{F} = (1/4)(31)$$

$$T_{\max} = 5.$$

Hence

$$\{\text{job } i \mid \Sigma p_j - d_i = 11 - d_i \leqslant T_{\max} = 5\} = \{4\}.$$

Thus schedule job 4 last, and find

$$\{\text{job } i \mid 11 - 2 - d_i \leqslant 5\} = \{2, 3\}.$$

Since $p_2 > p_3$, schedule job 2 in the third position, and find

$$\{\text{job } i \mid 9 - 3 - d_i \leqslant 5\} = \{1, 3\}.$$

Since $p_1 > p_3$, the new improved sequence is 3,1,2,4 with

$$\bar{F} = (1/4)(28)$$

$$T_{\max} = 5.$$

Smith's procedure could not be applied to this problem since $T_{\max} > 0$.

The new procedure can also be adapted to minimize weighted flow time in the fashion discussed by Smith [3].

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